HW8 Solutions

Dilip Raghavan

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1 Section 7.2

Problem 4. Consider the amount of money given away in the tournament before the last round. This consists of two subtournaments with $\frac{n}{2}$ players each. So $2a_{\frac{n}{2}}$ is given away prior to the last round. Now the subtournament won by the winner of the last round is just the whole tournament, which has n players. But we have already accounted for $\frac{n}{2}$ of them so far (as part of one of the two subtournaments prior to the final). So the winner of the last round gets $100\frac{n}{2}$ in the last round. So $a_n = 2a_{\frac{n}{2}} + 100\frac{n}{2} = 2a_{\frac{n}{2}} + 50n$, with the intial condition $a_2 = 300$ (if there are just 2 players, the winner gets 200 and the looser gets 100). By the table on page 292, the solution has the form $a_n = 50n(\log_2 n + A)$.

Problem 5. To merge 2 ordered k tuples into an ordered $2k$ tuple requires $2k - 1$ comparisons. So given a set of size n (where n is some power of 2), divide it into two sets of size $\frac{n}{2}$ each. Now sorting each half requires $a_{\frac{n}{2}}$ comparisons. So a total of $2a_{\frac{n}{2}}$ comparisons. Now, we have 2 ordered $\frac{n}{2}$–tuples which we would like to merge. This needs $n - 1$ comparisons. So $a_n = 2a_{\frac{n}{2}} + n - 1$, with the initial condition $a_2 = 1$.

2 Section 7.3

Problem 5. The recurrence relation is $a_n = 2a_{n-1} + a_{n-2}$, with initial conditions $a_1 = 2$ and $a_2 = 5$. From the recurrence relation we can compute $a_0 = 1$. Now, the characteristic polynomial is $\alpha^2 - 2\alpha - 1 = 0$. The roots are $\alpha = \frac{2\pm\sqrt{8}}{2}$, which simplifies to $1 \pm \sqrt{2}$. So $a_n = A\left(1 + \sqrt{2}\right)^n + B\left(1 - \sqrt{2}\right)^n$. Using the initial conditions $a_0 = 1$ and $a_1 = 2$, we get the equations $A + B = 1$ and $2 = A\left(1 + \sqrt{2}\right) + B\left(1 - \sqrt{2}\right)$. Solving the equations gives us $A = \frac{1+\sqrt{2}}{2\sqrt{2}}$ and $B = \frac{1-\sqrt{2}}{-2\sqrt{2}}$.

Problem 6. If the first chip is not red, then there are two possibilities for it: white or blue. Then there are a_{n-1} possibilities for what follows. If the first chip is red, then the second one cannot be red. So there are two possibilities for the second chip, and then a_{n-2} possibilities for the remaining. So $a_n = 2a_{n-1} + 2a_{n-2}$. The initial conditions are $a_1 = 3$ and $a_2 = 8$. Using the recurrence relation we can calculate $a_0 = 1$. The characteristic polynomial is $\alpha^2 - 2\alpha - 2 = 0$. The roots are $\alpha = \frac{2\pm\sqrt{12}}{2}$, which simplifies to $1 \pm \sqrt{3}$. So $a_n = A\left(1 + \sqrt{3}\right)^n + B\left(1 - \sqrt{3}\right)^n$. Using the initial conditions, we get the equations $A + B = 1$ and $A\left(1 + \sqrt{3}\right) + B\left(1 - \sqrt{3}\right) = 3$. Solving, we get $A = \frac{2+\sqrt{3}}{2\sqrt{3}}$, and $B = \frac{2-\sqrt{3}}{-2\sqrt{3}}$.

Problem 7. This year's change is $p_n - p_{n-1}$, and last year's change is $p_{n-1} - p_{n-2}$. So we have the equation $p_n - p_{n-1} = 2(p_{n-1} - p_{n-2})$, which simplifies to $p_n = 3p_{n-1} - 2p_{n-2}$. The characteristic polynomial is $\alpha^2 - 3\alpha + 2 = 0$. The roots are $\alpha = 2,\ 1$. So $p_n = A \cdot 2^n + B$. Using the initial conditions, we get the equations $B + A = 1$ and $B + 2A = 4$. The solutions are $A = 3$ and $B = -2$. So $p_n = 3 \cdot 2^n - 2$.

3 Section 7.4

Problem 8. (a) If the money is withdrawn only at the end of the year, the interest is paid on however much is left at the end of the previous year, which is a_{n-1}. So the recurrence relation is $a_n = 1.05a_{n-1} - 10n$, with $a_0 = 1000$. The solution to the homogenous part is $A \cdot (1.05)^n$. We look for a particular solution of the form $p(n) = Bn + C$. Substituting into the recurrence and solving for B and C we get $p(n) = 200n + 4200$. So $a_n = A \cdot (1.05)^n + 200n + 4200$. Using the initial condition, we get that $A = -3200$.

(b) If the money is withdrawn at the beginning of the year, the interest is earned only on the balance from the previous year minus $10n$. So $a_n = 1.05(a_{n-1} - 10n) = 1.05a_{n-1} - 10.5n$. The initial condition is $a_0 = 1000$. The homogeneous solution is once again $A \cdot (1.05)^n$. Now, again, we look for a particular solution of the form $p(n) = Bn + C$. Substituting and solving, we get $B = 210$ and $C = 4410$. So $a_n = A \cdot (1.05)^n + 210n + 4410$. Using the initial condition, we get $A = -3410$.

Problem 12. If the first slot is not 2, then there are 2 possibilities for it, and a_{n-1} possibilities for what follows. If the first slot is 2, then there can be no 1 in the sequence that follows. On the other hand, it can be *any* sequence of 0s and 2s. So there are 2^{n-1} possibilites for what comes afterwards. So $a_n = 2a_{n-1} + 2^{n-1}$. The inital condition is $a_1 = 3$. The solution to the homogenous part is $A \cdot 2^n$. Now, we *cannot* look for a particular solution of the form $B \cdot 2^n$ because that is already part of the homogeneous solution. So we must look for a particular solution $p(n) = B \cdot n \cdot 2^n$. Substituting we get,

$$B \cdot n \cdot 2^n = 2 \cdot B(n-1)2^{n-1} + 2^{n-1}$$

Dividing by 2^{n-1}, we get

$$2 \cdot B \cdot n = 2 \cdot B \cdot n - 2B + 1$$

So $B = \frac{1}{2}$. So we have $a_n = A \cdot 2^n + \frac{1}{2}n2^n$. Now, using the initial condition, we get the equation $3 = 2A + 1$. So $A = 1$.