# HW7 Solutions

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## 1 Section 6.4

**Problem 12.** Let us first see what the generating function would be if we treated the first two boxes as one box. Since $b_1 < b_2 \leq 4$, the minimum value for $b_1 + b_2$ is 1 and the maximum value is 7. So if the first two boxes were combined together into one box, the generating function would be

$$\left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}\right)\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)^3$$

Now, if $b_1 + b_2$ is equal to, for example, 5, we need to figure out how many ways there are to distribute 5 distinct balls into boxes 1 and 2 with $b_1 < b_2 \leq 4$. This is the same as deciding which balls will go into box 1. We could have 0, 1 or 2 balls in box 1. So there are $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}$ ways to distribute 5 distinct balls into boxes 1 and 2 with $b_1 < b_2 \leq 4$. So in the first factor above, the co-efficient of $\frac{x^5}{5!}$ should be $\left(\binom{5}{0}+\binom{5}{1}+\binom{5}{2}\right)$ instead of just 1. Doing this with all the co-efficients, we get the exponential generating function to be

$$\left(\binom{1}{0}\frac{x}{1!}+\binom{2}{0}\frac{x^2}{2!}+\left[\binom{3}{0}+\binom{3}{1}\right]\frac{x^3}{3!}+\left[\binom{4}{0}+\binom{4}{1}\right]\frac{x^4}{4!}+\left[\binom{5}{0}+\binom{5}{1}+\binom{5}{2}\right]\frac{x^5}{5!}+\left[\binom{6}{0}+\binom{6}{1}+\binom{6}{2}\right]\frac{x^6}{6!}+\right.$$
$$\left.+\left[\binom{7}{0}+\binom{7}{1}+\binom{7}{2}+\binom{7}{3}\right]\frac{x^7}{7!}\right)\left(\frac{x^0}{0!}+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots\right)^3$$

**Problem 14.** Let us fix $n$ and $m$. Let $a_r$ be the number of ways to distribute $r$ distinct balls into $n$ distinct boxes with exactly $m$ non–empty boxes. First consider the exponential generating function for the number of ways to distribute $r$ distinct balls into $m$ distinct boxes with no box empty. This is given by

$$\left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)^m$$

Now, if we have $n$ distinct boxes and we have to make sure that there are exactly $m$ non-empty boxes, then this is the same as first choosing $m$ out of the $n$ boxes and then distributing the $r$ balls into those $m$ boxes with none of them being empty. There are $\binom{n}{m}$ ways to choose the $m$ non–empty boxes out of the $n$. So the required generating function is

$$\binom{n}{m}\left(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\right)^m$$

## 2 Section 7.1

**Problem 17.** If she orders anything other than chicken salad on the first day, then in the remaining $n - 1$ days she needs to order lunch in such a way that she does not get chicken salad thrice in a row. So there are $2a_{n-1}$ possibilities for this case. Next, we consider two cases: Case I: she orders chicken salad the first day, but not the second; Case II: she orders chicken salad the first two days. For Case I we have 2 possibilities for the second day and then $a_{n-2}$ ways to order lunch for the remaining $n - 2$ days. For Case II, we have 2 possibilities for the third day and then $a_{n-3}$ ways to order lunch for the remaining $n - 3$ days. So the recurrence relation is $a_n = 2a_{n-1} + 2a_{n-2} + 2a_{n-3}$. Now, for the initial conditions, we have $a_1 = 3$, $a_2 = 9$ and $a_3 = 26$.

**Problem 20.** (a) If the arrangement starts with a red or gold flag then, we are left with arranging flags on a flagpole that is $n - 1$ feet high. There are $a_{n-1}$ ways of doing this. If the arrangement starts with a green flag, then we must arrange flags on a flagpole that is $n - 2$ feet high. There are $a_{n-2}$ ways of doing this. So $a_n = 2a_{n-1} + a_{n-2}$. $a_1 = 2$ and $a_2 = 5$

(b) If the arrangement starts with a green flag, then we are left with arranging flags on a pole that is $n - 2$ feet high subject to the requirement that there be no three 1 foot flags in a row. So there are $a_{n-2}$ ways of doing this. Now suppose the arrangement starts with a 1 foot flag. We have 2 choices for this first flag (red or gold). Now we are left with a pole that is $n - 1$. There are a total of $a_{n-1}$ ways to arrange flags on this pole so that there is no sequence of three 1 foot flags in a row. However, we want to exclude those sequences that start with two one foot flags. If the arrangement starts with two 1 foot flags, then the third flag must be a green flag. So there are $4a_{n-5}$ bad cases that we want to exclude. So there are $2(a_{n-1} - 4a_{n-5})$. So $a_n = a_{n-2} + 2a_{n-1} - 8a_{n-5}$.

(c) If the arrangement does not start with a green flag, then there are 2 possibilites for the first flag and then $a_{n-1}$ possibilities for what come afterwards. If the arrangement starts with a green flag, then out of the total $a_{n-2}$ possibilities for what comes afterwards, we must exclude the ones that start with gold followed by red. There are $a_{n-4}$ such arrangements. So $a_n = 2a_{n-1} + a_{n-2} - a_{n-4}$

**Problem 23.** If the sequence begins with 1, then the remaining sequence of $n-1$ things can be *any* quaternary sequence; so there are $4^{n-1}$ possibilities for this case. If it doesn't begin with 1, then there are 2 possibilities for the first spot (it can't begin with 0). Then there are $a_{n-1}$ possibilities for the remaining sequence of $n - 1$ things. So the recurrence relation is $a_n = 4^{n-1} + 2a_{n-1}$. For the initial conditions, $a_1 = 1$.

**Problem 27.** The first person can be paired off with any of the other $2n - 1$ people. Then the remaining $2n - 2$ people can be paired off in $a_{n-1}$ ways. So the recurrence relation is $a_n = (2n - 1)a_{n-1}$. For the initial conditions, $a_1 = 1$.

**Problem 34.** (a) Let $a_n$ be the number of quaternary sequences with an even number of 0s. If the first slot is not a 0, there are 3 choices for the first slot and $a_{n-1}$ choices for what come afterwards. If the first slot is a 0, then we need an odd number of 0s in what follows. There are a total of $4^{n-1}$ quaternary sequences of length $n - 1$. Of these $a_{n-1}$ have an even number of 0s. So $4^{n-1} - a_{n-1}$ have an odd number of 0s. So $a_n = 3a_{n-1} + 4^{n-1} - a_{n-1} = 2a_{n-1} + 4^{n-1}$. So you just need one recurrence relation for this part. The initial condition is $a_1 = 3$.

(b) If the first slot is neither 0 nor 1, then there are 2 possibilities for the first slot and $a_{n-1}$ possibilities for what follows. If the first slot is either 0 or 1, then there are two possibilities for the first slot and $4^{n-1} - a_{n-1}$ possibilities for what follows. So $a_n = 2a_{n-1} + 2(4^{n-1} - a_{n-1}) = 2 \times 4^{n-1}$

(c) Now we need three sequences. Let $a_n$ be the number of $n$–digit quaternary sequences with an even number of 0s and an even number of 1s. Let $b_n$ be the number of $n$–digit quaternary sequences with an odd number of 0s and and even number of 1s, and let $c_n$ be the number of $n$–digit quaternary sequences with an even number of 0s and an odd number of 1s. Now, to get $a_n$, if the sequences begins with anything other than 0 or 1, there are two possibilities for the first slot and then $a_{n-1}$ possibilities for what remains. If the sequence begins with 0, then there are $b_{n-1}$ possibilities for what remains. If the sequence begins with 1, there are $c_{n-1}$ possibilities for what remains. So $a_n = 2a_{n-1}+b_{n-1}+c_{n-1}$. Now notice that $4^n-a_n-b_n-c_n$ is equal to the number of quaternary sequences with an odd number of 0s and an odd number of 1s. So following a similar analysis we get $b_n = 2b_{n-1}+a_{n-1}+4^{n-1}-a_{n-1}-b_{n-1}-c_{n-1}$ and $c_n = 2c_{n-1} + 4^{n-1} - a_{n-1} - b_{n-1} - c_{n-1} + a_{n-1}$