

HW6 Solutions

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1 Section 6.2

Problem 23. Let a_n denote the number of ways to choose n boxes of candies with at least one but not more than 5 boxes of each type. The generating function for the sequence of a_n is

$$(x + x^2 + x^3 + x^4 + x^5)^7$$

Since we want to choose 300 candies, we want to select 15 boxes of candy. So we want the co-efficient of x^{15} in the above polynomial. Factoring out x , we have

$$(x + x^2 + x^3 + x^4 + x^5)^7 = x^7 [1 + x + x^2 + x^3 + x^4]^7$$

So we want the co-efficient of x^8 in $[1 + x + x^2 + x^3 + x^4]^7$. We use the identity for $\frac{1 - x^5}{1 - x}$

$$\begin{aligned} [1 + x + x^2 + x^3 + x^4]^7 &= \left[\frac{1 - x^5}{1 - x} \right]^7 = \left(\binom{7}{0} - \binom{7}{1}x^5 + \binom{7}{2}x^{10} - \binom{7}{3}x^{15} + \dots \right) \\ &\quad \left(\binom{0+7-1}{0} + \binom{1+7-1}{1}x + \binom{2+7-1}{2}x^2 + \dots \right) \end{aligned}$$

Since we want the co-efficient of x^8 , the answer is $\binom{7}{0} \binom{8+7-1}{8} - \binom{7}{1} \binom{3+7-1}{3}$.

Problem 26. Let us first count the number of ways to divide the objects into ordered piles. We can then divide that by 2 to get the answer. This works because the two piles cannot be the same. This is because there are an odd number of objects of each kind. So it is not possible for both piles to have the same number of, say, apples. If there were an even number of objects of each kind, then we would have to consider the case when both piles were the same, and so we cannot just divide the answer by 2.

Counting the number of ways to divide into two *ordered* piles is the same as the number of ways to select 15 objects from 6 types of objects with at most 5 objects of each type. Let a_n be the number of ways to select n such objects. The generating function for this sequence is

$$(x^0 + x^1 + x^2 + x^3 + x^4 + x^5)^6$$

Now, we want the co-efficient of x^{15} in this polynomial. We have

$$\begin{aligned} (x^0 + x^1 + x^2 + x^3 + x^4 + x^5)^6 &= \left[\frac{1 - x^6}{1 - x} \right]^6 = \left(\binom{6}{0} - \binom{6}{1}x^6 + \binom{6}{2}x^{12} + \dots \right) \\ &\quad \left(\binom{0+6-1}{0}x^0 + \binom{1+6-1}{1}x^1 + \binom{2+6-1}{2}x^2 + \dots \right) \end{aligned}$$

So the answer is

$$\frac{\binom{6}{0}\binom{15+6-1}{15} - \binom{6}{1}\binom{9+6-1}{9} + \binom{6}{2}\binom{3+6-1}{3}}{2}$$

Problem 29. First distribute the steaks and then the lobsters. Let a_n be the number of ways to distribute n steaks to 4 people with each person getting at most 5 steaks. The generating function is

$$(x^0 + x + x^2 + x^3 + x^4 + x^5)^4$$

We want the co-efficient of x^{10} . We have

$$(x^0 + x + x^2 + x^3 + x^4 + x^5)^4 = \left[\frac{1 - x^6}{1 - x} \right]^4$$

$$\left(\binom{4}{0} - \binom{4}{1}x^6 + \dots \right) \left(\binom{0+4-1}{0} + \binom{1+4-1}{1}x + \binom{2+4-1}{2}x^2 + \binom{3+4-1}{3}x^3 + \binom{4+4-1}{4}x^4 + \dots \right)$$

Since we want the co-efficient of x^{10} , the number of ways to distribute the steaks is

$$\binom{4}{0}\binom{10+4-1}{10} - \binom{4}{1}\binom{4+4-1}{4}$$

Now to distribute the lobsters, let a_n be the number of ways to distribute n lobsters to 4 people with each person getting at most 5 lobsters. The generating function is the same as above, but now, we want the co-efficient of x^{15} . So the final answer is

$$\left[\binom{4}{0}\binom{10+4-1}{10} - \binom{4}{1}\binom{4+4-1}{4} \right] \left[\binom{4}{0}\binom{15+4-1}{15} - \binom{4}{1}\binom{9+4-1}{9} + \binom{4}{2}\binom{3+4-1}{3} \right]$$

2 Section 6.3

Problem 5. We are looking for all solutions to the equation $e_1 + 5e_2 + 10e_3 + 25e_4 = r$. So the generating function is

$$(x^0 + x^1 + x^2 + x^3 + \dots)(x^{5 \cdot 0} + x^{5 \cdot 1} + x^{5 \cdot 2} + x^{5 \cdot 3} + \dots)(x^{10 \cdot 0} + x^{10 \cdot 1} + x^{10 \cdot 2} + x^{10 \cdot 3} + \dots)(x^{25 \cdot 0} + x^{25 \cdot 1} + x^{25 \cdot 2} + x^{25 \cdot 3} + \dots)$$

This is equal to

$$\frac{1}{1-x} \frac{1}{1-x^5} \frac{1}{1-x^{10}} \frac{1}{1-x^{25}}$$

Problem 22. Break the counting into 2 cases: either the $(n+1)^{\text{th}}$ element is by itself or it is in a subset that contains some of the other n elements. For the first case, if the $(n+1)^{\text{th}}$ element is by itself, then we just need to partition the first n elements into $k-1$ non-empty subsets: $\left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\}$ ways. For the other case, the $(n+1)^{\text{th}}$ element is not in a subset by itself. So first partition the first n elements into k non-empty subsets: $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ ways. Next we must decide which of these subsets to put the $(n+1)^{\text{th}}$ element into. Since we have k non-empty subsets, we have k choices for this. So the total for this case is $k \times \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$. Adding the two cases we have $\left\{ \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\} + k \times \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$.

3 Section 6.4

Problem 6. Let a_n be the number of ways to distribute n toys among 4 children with the first child getting at least 2 toys. The exponential generating function is

$$\left(\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots\right)^3$$

This is equal to $(e^x - x - 1)e^{3x} = e^{4x} - xe^{3x} - e^{3x}$. Using the Taylor series for e^x this is equal to

$$\sum_{n=0}^{\infty} \frac{4^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{3^n x^{n+1}}{n!} - \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}.$$

So the co-efficient of $\frac{x^8}{8!}$ is $4^8 - 8 \times 3^7 - 3^8$.

Problem 10. (a) Let a_r be the number of r -digit ternary sequences in which no digit occurs exactly twice. The exponential generating function is

$$\left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^3$$

This is equal to $(e^x - \frac{x^2}{2})^3 = e^{3x} - \frac{3}{2}e^{2x}x^2 + \frac{3}{4}e^x x^4 - \frac{x^6}{8}$. Using the Taylor series for e^x this is

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{3}{2} \frac{2^n x^{n+2}}{n!} + \sum_{n=0}^{\infty} \frac{3}{4} \frac{x^{n+4}}{n!} - \frac{x^6}{8}$$

Therefore, if $r \neq 6$, then the co-efficient of $\frac{x^r}{r!}$ is $3^r - \frac{3}{2}2^{r-2}(r-1)(r) + \frac{3}{4}(r-3)(r-2)(r-1)(r)$. If $r = 6$, then it is $\left[3^r - \frac{3}{2}2^{r-2}(r-1)(r) + \frac{3}{4}(r-3)(r-2)(r-1)(r)\right] - \frac{6!}{8}$.

(b) Let a_r denote the number of r -digit ternary sequences in which 0 and 1 each appear a positive even number of times. The exponential generating function is

$$\left(\frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)^2 \left(\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

This is equal to $\left(\frac{e^x + e^{-x} - 2}{2}\right)^2 e^x = \frac{e^{3x} - 4e^{2x} + 6e^x + e^{-x} - 4}{4}$. Using the Taylor series for e^x this is

$$\sum_{n=0}^{\infty} \frac{1}{4} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} 4 \frac{1}{4} \frac{2^n x^n}{n!} + \sum_{n=0}^{\infty} 6 \frac{1}{4} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{4} \frac{(-1)^n x^n}{n!} - \frac{4}{4}$$

So the co-efficient of $\frac{x^r}{r!}$ is $\frac{3^r - 4 \times 2^r + 6 + (-1)^r}{4}$