HW5 Solutions

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1 Section 6.1

Problem 4. (a) We want all solution to the equation $e_1 + e_2 + e_3 + e_4 + e_5 = r$ with $0 \le e_i \le 5$ So the generating function is

$$(1 + x + x^2 + x^3 + x^4 + x^5)^5$$

(b) We want all solutions to the equation $e_1 + e_2 + e_3 + e_4 = r$ with $3 \le e_i \le 6$. So the generating function is

$$(x^3 + x^4 + x^5 + x^6)^4$$

(c) We want all solutions to the equation $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 = r$ with $e_i \ge 1$. So the generating function is

$$\left(\sum_{k=1}^{\infty} x^k\right)^7 = \left[\frac{x}{1-x}\right]^7$$

(d) We want solutions to $e_1 + e_2 + e_3 = r$ with $e_1 \le 5$. So the generating function is

$$(1 + x + x^2 + x^3 + x^4 + x^5) \times \left(\sum_{k=0}^{\infty} x^k\right)^2 = (1 + x + x^2 + x^3 + x^4 + x^5) \times \left[\frac{1}{1-x}\right]^2$$

Problem 14. (a) We want solutions to the equation $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = r$ with e_1, e_2 and e_3 odd and the e_4, e_5 and e_6 even. So the generating function is

$$(x + x^3 + x^5)^3(x^2 + x^4 + x^6)^3$$

(b) We want solutions to $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = r$ with $e_i \ne i$. So the generating function is

$$(x^2 + x^3 + x^4 + x^5 + x^6)(x + x^3 + x^4 + x^5 + x^6)(x + x^2 + x^4 + x^5 + x^6)$$
$$(x + x^2 + x^3 + x^5 + x^6)(x + x^2 + x^3 + x^4 + x^6)(x + x^2 + x^3 + x^4 + x^5)$$

Problem 16. We want solutions to the equation $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 = r$ with $0 \le e_i \le 9$. So the generating function is

$$\left[(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)\right]^6$$

Problem 21. You can't have a variable number of factors. So you *cannot* have r in the power.

Problem 23. Let e_1 denote number of pennies used, e_2 the number of nickels used and e_3 the number of dimes used. So we want solutions to the equation $e_1 + 5e_2 + 10e_3 = r$. So the factor corresponding to nickels can only contribute powers that are multiples of 5 and the factor corresponding to dimes can only contribute powers that are multiples of 10. So the generating function is

$$(x^0 + x^1 + x^2 + x^3 + \cdots)(x^{5\cdot 0} + x^{5\cdot 1} + x^{5\cdot 2} + x^{5\cdot 3} + \cdots)(x^{10\cdot 0} + x^{10\cdot 1} + x^{10\cdot 2} + x^{10\cdot 3} + \cdots)$$

Problem 24. (a) Each of the 50 states (since the book is by a US author, I'm assuming it's talking about US states) can contribute either 0 contestants or 1 contestant. So we make one factor for each state, and each factor has a x^0 term and a x^1 term. Now if a factor contributes a x^1 term, since there are 5 ways to choose one contestant from the state, it contributes 5 to the total number of ways to choose 20 contestants. So we want the co-efficient of x^{20} in

$$(1 + 5x)^{50}$$

(b) Now each factor can contribute x^0, x^1, x^2 or x^3. If it contributes x^i then it contributes $\binom{5}{i}$ to the total number of ways. So we want the co–efficient of x^{20} in

$$\left[\binom{5}{0}1 + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3\right]^{50}$$

2 Section 6.2

Problem 13. The smallest non–zero co–efficient is $(x^2)^5 = x^{10}$. So the co-eeficient of x^9 is 0.

Problem 19. We want the co–efficient of x^{12} in

$$(1 + x + x^2 + x^3 + x^4)^5$$

We use the fact that

$$(1 + x + x^2 + x^3 + x^4) = \frac{1 - x^5}{1 - x}.$$

So we want the co-efficient of x^{12} in $\frac{(1-x^5)^5}{(1-x)^5}$. Using the binomial theorem and the identity for $\left(\frac{1}{1-x}\right)^5$ we have,

$$(1 - x^5)^5 = \binom{5}{0} - \binom{5}{1}x^5 + \binom{5}{2}x^{10} + \cdots$$

$$\left(\frac{1}{1-x}\right)^5 = \binom{0+5-1}{0} + \binom{1+5-1}{1}x + \binom{2+5-1}{2}x^2 + \binom{3+5-1}{3}x^3 + \cdots$$

Since in the first polynomial the power x becomes greater than 12 after the first three terms, we just need to consider the products of those three terms with the terms in the second polynmial. So there are 3 ways to get x^{12}: $x^0 \times x^{12}$, $x^5 \times x^7$ and $x^{10} \times x^2$. So the answer is

$$\binom{5}{0} \times \binom{12+5-1}{12} - \binom{5}{1} \times \binom{7+5-1}{7} + \binom{5}{2} \times \binom{2+5-1}{2}$$