

HW4 Solutions

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1 Section 5.4

Problem 48. Let us first make an inventory of everything we have. The vowels: one I, one U and one O. The consonants: one N, one S, two Ts, two Rs and one C. Now, we know that the vowels must occur in alphabetical order, so let's first put them down:

I O U

We also know that we must begin or end with a block of 2 Ts. We must decide whether we begin or end with them: 2 choices.

TT I O U

We are left with 5 letters to arrange. Let us first think about how many ways there are to distribute 5 empty slots among the letters we have arranged so far. We will then count the number of ways to arrange the letters that are left over in those empty slots. So we want to distribute 5 identical empty slots into 4 boxes:

TT $\text{box } 1$ I $\text{box } 2$ O $\text{box } 3$ U $\text{box } 4$

Since we must have at least 2 consonants between any two vowels, we must make sure that there are at least 2 empty slots in each of boxes 2 and 3. So put two slots into each of those boxes. We have just one slot left, which we must distribute into 4 boxes: 4 ways. Now we must decide how the letters are placed in the slots:

TT — I — — O — — U

Since there are two identical Rs, there are $\frac{5!}{2!}$ ways to do this. So the answer is $2 \times 4 \times \frac{5!}{2!}$

Problem 50. Again, let us begin by making an inventory of everything we have. The vowels: two Es, two Is. The consonants: one R, one V, one S, one T and one D.

- (a) Let us count the complement, which is the number of arrangements with vowels in increasing order. Firstly, the total number of arrangements is $\frac{9!}{2!2!}$. Now if the vowels are in increasing order, let us first write them down

E E I I

Again, let us first distribute 5 empty slots for the 5 remaining letters into 5 boxes: $\binom{5+5-1}{5}$ ways. Now we need to place the letter in those slots: 5! ways. So the answer is $\frac{9!}{2!2!} - \left[\binom{5+5-1}{5} \times 5! \right]$ (You could also do this problem by first arranging the consonants and then distributing slots for vowels).

(b) Let us first arrange the vowels and then we will distribute slots for the consonants. But we must break up into four cases depending on whether the arrangement of the vowels has consecutive Es and Is or not.

- Case I: no consecutive Es or Is. This means there is an I between the two Es and an E between the 2 Is. There are 2 such arrangements of the Es and Is. For example, such an arrangement might look like:

E I E I

Now, we are free to distribute 5 slots for the 5 consonants in any way we please into 5 boxes: $\binom{5+5-1}{5}$ ways. Next we place the consonants in these 5 slots: $5!$ ways. So for this case we have $2 \times \binom{5+5-1}{5} \times 5!$ possibilities.

- Case II: consecutive Es but not consecutive Is. There is only one such arrangement of the vowels:

I E E I

Now, we can distribute 5 slots for the 5 remaining letters in 5 boxes making sure that the box between the two Es is not empty: $\binom{4+5-1}{4}$ ways. Finally, there are $5!$ ways to place the 5 consonants in these 5 slots. So for this case we have $\binom{4+5-1}{4} \times 5!$ ways.

- Case III: consecutive Is but not consecutive Es. This is just like Case II with I and E switched. So we have $\binom{4+5-1}{4} \times 5!$ possibilities for this case too.
- Case IV: both Is and Es are consecutive. There are 2 such arrangements of the vowels. For example:

I I E E

Now we distribute 5 slots for the 5 consonants into 5 boxes, but we must make sure the both the box between the Is and the box between the Es are non-empty: $\binom{3+5-1}{3}$ ways. Finally, there are $5!$ ways to arrange the consonants in those slots. So for this case we have $2 \times \binom{3+5-1}{3} \times 5!$ ways.

So the final answer is $2 \times \binom{5+5-1}{5} \times 5! + 2 \times \binom{4+5-1}{4} \times 5! + 2 \times \binom{3+5-1}{3} \times 5!$

(c) Again we can count the complement. We know the total number of arrangements without consecutive Es or Is from part (b). From this we must subtract the number of such arrangements where the vowels *are* in increasing order.

Let us first put down the vowels:

E E I I

Now, we distribute 5 slots for the 5 consonants into 5 boxes making sure that the box between the Es and the box between the Is is non-empty: $\binom{3+5-1}{3}$ ways. Now there are $5!$ ways to arrange the 5 consonants in these 5 slots. So the answer is

$$\left[2 \times \binom{5+5-1}{5} \times 5! + 2 \times \binom{4+5-1}{4} \times 5! + 2 \times \binom{3+5-1}{3} \times 5! \right] - \left[\binom{3+5-1}{3} \times 5! \right]$$

Problem 60. Break up into cases depending on how many clusters of α s there are and how many α s per cluster. There are seven cases, though each case is easy to handle by itself!

- Case I: 4 clusters of 2 α s each

$$\alpha\alpha \quad \alpha\alpha \quad \alpha\alpha \quad \alpha\alpha$$

- Case II: 2 clusters of threes and one cluster of 2 (keep in mind that there are 3 ways to arrange the clusters)

$$\alpha\alpha\alpha \quad \alpha\alpha\alpha \quad \alpha\alpha$$

- Case III: one cluster of 4 and 2 clusters of 2 each

$$\alpha\alpha\alpha\alpha \quad \alpha\alpha \quad \alpha\alpha$$

- Case IV: one cluster of 5 and a cluster of 3

$$\alpha\alpha\alpha\alpha\alpha \quad \alpha\alpha\alpha$$

- Case V: one cluster of 6 and a cluster of 2

$$\alpha\alpha\alpha\alpha\alpha\alpha \quad \alpha\alpha$$

- Case VI: one cluster of 8

$$\alpha\alpha\alpha\alpha\alpha\alpha\alpha\alpha$$

- Case VII: two clusters of 4 (keep in mind there is just one way to arrange the two clusters)

$$\alpha\alpha\alpha\alpha \quad \alpha\alpha\alpha\alpha$$

Problem 63. Arrange m blocks of 01:

$$01 \quad 01 \quad 01 \quad 01 \cdots 01 \quad 01$$

Now, we have used up $2m$ out of the n spaces. We must decide what to do with the remaining $n - 2m$ spaces. We think of the all the places where we can put them as boxes:

$$\text{box } 1 \quad 01 \quad \text{box } 2 \quad 01 \quad \text{box } 3 \quad 01 \quad \text{box } 4 \quad 01 \cdots 01 \quad \text{box } m \quad 01 \quad \text{box } m+1$$

Now there are exactly m occurrences of 01. So 01 cannot occur in any box. This means that in any given box, all the 1s must occur on the left and all the 0s must occur on the right. So we can think of each of the boxes illustrated above as being split into two:

$$1s \mid 0s \quad 01 \quad 1s \mid 0s \quad 01 \quad 1s \mid 0s \quad 01 \quad 1s \mid 0s \quad 01 \cdots 01 \quad 1s \mid 0s \quad 01 \quad 1s \mid 0s$$

Thus each of the $m + 1$ boxes we had before has been divided into two. So now we have $2m + 2$ boxes. And we just need to decide how many spaces to put into each box. We don't need to decide whether a 1 or a 0 goes into a given space because that is determined by which box the space is in. So we need to distribute $n - 2m$ spaces into $2m + 2$ boxes. So the answer is $\binom{(n - 2m) + (2m + 2) - 1}{n - 2m} = \binom{n + 1}{n - 2m}$.

Problem 65. Decide exactly how many balls go in the first m boxes. This can be anything between s and r . Say we decide to put i balls into the first m boxes. There are $\binom{i+m-1}{i}$ ways to distribute those i balls into the m boxes. We are now left with $r-i$ balls to distribute into the last $n-m$ boxes. There are $\binom{(r-i)+(n-m)-1}{r-i}$ ways to do this. So there are $\binom{i+m-1}{i} \times \binom{(r-i)+(n-m)-1}{r-i}$ to distribute r balls into n boxes with the first m boxes having *exactly* i balls. Now we sum over all the possibilities for i . So the answer is

$$\sum_{i=s}^r \binom{i+m-1}{i} \times \binom{(r-i)+(n-m)-1}{r-i}$$

2 Section 5.5

Problem 9. (a) The left hand side is the number of ways to choose 2 things from a set of $2n$ things. For example, say we have n distinct apples and n distinct oranges, and suppose we want to pick a set of 2 fruits from among them. There are $\binom{2n}{2}$ ways to do this. We could also count the same thing by breaking into cases as follows. Case I: no apples are chosen; Case II: no oranges are chosen; Case III: one of each is chosen. There are $\binom{n}{2}$ ways to do case I, and the same for case II. For case III, there are n ways to choose an apple and n ways to choose an orange. So there are n^2 ways to choose one of each. Therefore, adding up the three cases, $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

(b) First of all notice that the product $\binom{n+r-1}{2r} \times \binom{2r}{r}$, which occurs on the right hand side, is equal to $\binom{n+r-1}{r} \times \binom{n-1}{r}$ (see example 1 of Section 5.5 in the book). So the right hand side is equal to $n \binom{n+r-1}{r} \binom{n-1}{r}$. Now since the term $\binom{n+r-1}{r}$ occurs both on the left and on the right, we just need to show that $(n-r) \binom{n}{r} = n \binom{n-1}{r}$.

Now, suppose we have an n element set. How many ways are there to choose one element from this set followed by a set of r other elements (all different from that one element that was first chosen)? We can count the number of possibilities in two different ways. First choose the one element: n ways. Next choose the set of r elements from what remains (since this set can't include the element chosen): $\binom{n-1}{r}$ ways. So the answer is $n \binom{n-1}{r}$. But we could also first choose the r element set: $\binom{n}{r}$ ways. We can then choose the one element from what remains (again to make sure it is not part of the r element set): $n-r$ ways. So the answer is also $\binom{n}{r} (n-r)$. So $\binom{n}{r} (n-r) = n \binom{n-1}{r}$.

Problem 11. (a) Say we have n distinct symbols and we have 3 slots in which to place them:

— — —

How many are there to do this if we are allowed to repeat symbols? Since we can put any of the n symbols in each slot, there are n^3 ways to do this. Let us count the same thing by breaking into cases depending on exactly how many symbols we use. Case I: we use exactly one symbol; Case II: we use exactly two symbols; Case III: we use exactly three symbols. For case I, we first choose the symbol

out of the n symbols: $\binom{n}{1}$ ways. There is just one way to arrange that one symbol in the three slots. So case I has $\binom{n}{1}$ possibilities. For case II, first choose the 2 symbols: $\binom{n}{2}$ ways. There are 6 ways to arrange the 2 symbols in the 3 slots. So we have $6 \times \binom{n}{2}$ for case II. For case III, there are $\binom{n}{3}$ ways to choose the 3 symbols and then $3! = 6$ ways to arrange the 3 symbols in the 3 slots. So $6 \times \binom{n}{3}$ possibilities for case III. Adding up the 3 cases, we see that $n^3 = \binom{n}{1} + 6 \times \binom{n}{2} + 6 \times \binom{n}{3}$.

(b) This is just like Example 5 in the book (see page 228).

Problem 19. On the left hand side we have $\left[\sum_{k=0}^n \binom{n}{k} \right]^2$, which is equal to $[2^n]^2 = 2^{2n}$. On the right hand side we have $\sum_{k=0}^{2n} \binom{2n}{k}$ which is equal to 2^{2n} .

Problem 26. We will show by a counting argument that

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k} = \binom{n+m}{m}$$

The left hand side counts the number of m element subsets of a $n+m$ element set. Let us take an $n+m$ element set which has been divided into two parts of size n and m . For concreteness, suppose we have n distinct apples and m distinct oranges. We want to choose a set of m fruits from among these. There are $\binom{n+m}{m}$ ways to do this. Now, let us count the same thing in a different way by breaking into cases depending on exactly how many apples we choose. Suppose we decide to choose exactly k apples. There are $\binom{n}{k}$ ways to choose the apples. Now, we must pick $m-k$ oranges. There are $\binom{m}{m-k}$ ways to do this. So there are $\binom{n}{k} \times \binom{m}{m-k}$ ways to choose m fruits, if we decide to pick exactly k apples. Since $n \leq m$, we can choose any number of apples from 0 to n . Summing over all the cases we see that

$$\binom{n+m}{m} = \sum_{k=0}^n \binom{n}{k} \times \binom{m}{m-k}$$

However, $\binom{m}{m-k} = \binom{m}{k}$. Therefore

$$\binom{n+m}{m} = \sum_{k=0}^n \binom{n}{k} \times \binom{m}{m-k} = \sum_{k=0}^n \binom{n}{k} \times \binom{m}{k}$$

Problem 32. Given a set X with n elements, we want to count the number of pairs of subsets $\langle a, b \rangle$, where a and b are disjoint subsets of X with $|a \cup b| = m$. We count the number of such pairs in two different ways. First we break into cases depending on the size of a . a can have anywhere between 0 and m elements. Once the size of a is fixed, the size of b is also determined, since they have to union up to a set of size m . Suppose we decide $|a| = k$. There are $\binom{n}{k}$ choices for a . Now b must be a set of size $m-k$ chosen from among the remaining $n-k$ things. So there are $\binom{n-k}{m-k}$ choices for b . So there are a total of $\binom{n}{k} \times \binom{n-k}{m-k}$ such pairs $\langle a, b \rangle$, if $|a| = k$. Now summing over all possibilities for k we get that the total number of such pairs $\langle a, b \rangle$ is

$$\sum_{k=0}^m \binom{n}{k} \times \binom{n-k}{m-k}$$

Now let us count the number of such pairs in a different way. Let us first choose a subset c of X with m elements and then let us partition it into a pair of disjoint sets $\langle a, b \rangle$, with $a \cup b = c$. There are $\binom{n}{m}$ ways to

choose the set c . Once c is chosen, we must how to partition it into a pair of disjoint sets $\langle a, b \rangle$. a can be any subset of c . So there are 2^m choices for a . But once a is fixed, b is also fixed. So there are 2^m ways to partition c . So the number of such pairs $\langle a, b \rangle$ is also equal to $\binom{n}{m} \times 2^m$.