HW3 Solutions

Dilip Raghavan

September 21, 2008

1 Section 5.3

Problem 11. First choose the 2 kinds of fruit. Since there are 5 kinds, there are $\binom{5}{2}$ ways to do this. Now we want to select 10 objects from 2 kinds of objects, making sure that there is atleast one of each kind (we can't have all the fruits be of the same kind because we need *exactly two* kinds). So we first select one fruit of each kind. Now we need to select the remaining 8 fruits from 2 kinds of fruit. There are $\binom{8+2-1}{8}$ ways to do this. So the answer is $\binom{5}{2} \times \binom{8+2-1}{8}$.

Problem 12. The best way to deal with the pink, lavender and tan balls is to break into cases depending on exactly how many of those balls we choose. So there are 4 cases.

- Case I: We don't choose any of the pink, lavender and tan balls. This means that we must select 8 balls from 3 kinds of balls (red, white and blue). There are $\binom{8+3-1}{8}$ ways to do this.
- Case II: We choose exactly one out of the pink, lavender and tan balls: $\binom{3}{1}$ ways. Now we choose 7 balls from 3 kinds: $\binom{7+3-1}{7}$ ways. So a total of $\binom{3}{1} \times \binom{7+3-1}{7}$ ways.
- Case III: Choose exactly two out of the pink, lavender and tan balls: $\binom{3}{2}$ ways. Now choose 6 balls from 3 kinds: $\binom{6+3-1}{6}$ ways. So the total is $\binom{3}{2} \times \binom{6+3-1}{6}$.
- Case IV: We choose all three of the pink, lavender and tan balls. Now choose 5 balls from 3 kinds: $\binom{5+3-1}{5}$ ways.

So the final answer is $\binom{8+3-1}{8} + \binom{3}{1} \times \binom{7+3-1}{7} + \binom{3}{2} \times \binom{6+3-1}{6} + \binom{5+3-1}{5}$

Problem 15. First choose 6 out of the 10 digits: $\binom{10}{6}$ ways. There are two cases to consider.

- Case I: One of the 6 digits is used thrice and the rest are used once each. First decide which digit is used thrice: 6 choices. There are $\frac{8!}{3!}$ ways to arrange these digits in a row (3! appears because there are 3 of the same kind). So the total for this case is $6 \times \frac{8!}{3!}$

- Case II: Two digits are used twice each and the rest once each. Decide which of the 6 digits will be used twice each: $\binom{6}{2}$ choices. There are $\frac{8!}{2!2!}$ arrangements of these. So the total is $\binom{6}{2} \times \frac{8!}{2!2!}$

So the answer is $\binom{10}{6} \times \left[6 \times \frac{8!}{3!} + \binom{6}{2} \times \frac{8!}{2!2!}\right]$.

Problem 16. The best way to do this problem is to break it up into cases. There are lots of different cases to consider. Firstly, there are 3 broad cases: one even, 2 odds; 2 evens, 4 odds; and 3 evens, 6 odds. Now each case must be broken into subcases depending on how many repetitions of digits there are.

2 Section 5.4

Problem 5. Let us first give each principal 2 doughnuts of each kind. We are left with 8 chocolate doughnuts, 2 cinnamon doughnuts and 6 powered sugar doughnuts. Let us distribute the remaining doughnuts one kind at a time. First distribute the chocolate doughnuts: $\binom{8+4-1}{8}$ ways. Next distribute the cinnamon doughnuts: $\binom{2+4-1}{2}$ ways. Finally distribute the powdered sugar doughnuts: $\binom{6+4-1}{6}$ ways. So the total is $\binom{8+4-1}{8} \times \binom{2+4-1}{2} \times \binom{6+4-1}{6}$.

Problem 23. We think of the runs of 1s and 0s as boxes. So we have 4 different boxes: The first run of 1s is Box 1, the first run of 0s is Box 2, the second run of 1s is Box 3 and the second run of 0s is Box 4. So we want to distribute 18 identical objects (the 18 slots for placing 0s and 1s) into 4 boxes such that no box is empty and such that at least one of Boxes 1 and 3 have at least 8 objects. Notice that there is only one outcome with both Boxes 1 and 3 having 8 objects. This is because Boxes 2 and 4 must have at least one object each. Let us first out one object into each of Boxes 2 and 4. We are left with 16 objects. To deal with the requirement that at least one of Boxes 1 and 3 must have at least 8 objects, let us first choose one of the 2 boxes: 2 ways to do this. Next let us throw in 8 objects into that box and one object into the other one (to make sure it not empty). We are left with 7 objects to distribute into 4 boxes: there are $\binom{7+4-1}{7}$ ways to do this. This way of counting has double counted the one outcome where there are 8 objects in both Box 1 and 3. So the total is $2 \times \binom{7+4-1}{7} - 1$.

Problem 38. Put $y_i = x_i + 10$. We want the number of solutions to the inequality $y_1 + y_2 + y_3 + y_4 \leq 55$ with $y_i \geq 0$. For a fixed value n, the number of solutions to $y_1 + y_2 + y_3 + y_4 = n$ with $y_i \geq 0$ is $\binom{n+4-1}{n}$. So the answer is:

$$\sum_{n=0}^{n=55} \binom{n+4-1}{n}.$$

Another way to deal with the fact that we have an inequality instead of an equality is to introduce a fifth variable, say y_5. Now, we want to count the number of solutions to the *equation* $y_1+y_2+y_3+y_4+y_5 = 55$. We are thinking of distributing 55 objects into 5 boxes. When we do this, the first four boxes will have *atmost* 55 objects, and we think of the "leftovers" as going into the fifth box. So the number of solutions to the equation $y_1 + y_2 + y_3 + y_4 + y_5 = 55$ is the same as the number of solutions to the inequality $y_1 + y_2 + y_3 + y_4 \leq 55$. So the answer is $\binom{55+5-1}{55}$. This is the same as the sum given above.