HW2 Solutions

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1 Section 5.2

Problem 28. Though the problem does not make this clear, I'm assuming that the teams are not ordered. That is, if I take all the members of Team 1 and swap them with all the members of Team 2, then that counts as the same outcome. Let us first count the outcomes as if the teams were ordered. Then in the end we can divide the answer by two.

Count the complement – the number of ways to split into two teams where all the women end up on one of the teams. Let us first count all ways to split into two teams without any restrictions. Note that once the members of Team 1 are determined, Team 2 are also determined. So it is enough to choose the members of Team 1. There are $\binom{8}{4}$ ways to do this. Now if all the women end up on one of the two teams, we must first decide which team will the contain all the women. There are 2 ways to do this (either Team 1 or Team 2). Once this has been decided we have choose one man for that team. There are 5 ways to do this. This determines all the members of that team. But then the other is also determined (they are the ones left over). So there are 2×5 ways to split into two teams so that all the women end up on the same team. What we are looking for is the complement: $\binom{8}{4} - 2 \times 5$.

Finally, since we are assuming the teams are not ordered, we divide the answer by 2. So $\frac{1}{2}\left[\binom{8}{4} - 2 \times 5\right]$ is the answer.

Problem 43. The total number of five card hands is $\binom{52}{5}$.

(a) There are two cases: either all five cards are among Ace, King, Queen and Jack, or there is one other value.

- Case I: all cards are among the four values. In this case, there are 2 cards of the same value and one from each of the other three. So first choose the value with 2 cards. There are 4 ways to do this. Next choose 2 cards of that value: $\binom{4}{2}$ ways. Now choose one from each of the other three values: $\binom{4}{1}^3$. So the total is $4 \times \binom{4}{2} \times \binom{4}{1}^3$.
- Case II: there is one other value. Choose that value. There are $13 - 4$ ways to do this. Next choose one card from each of the 5 values: $\binom{4}{1}^5$ ways. So total is $(13 - 4) \times \binom{4}{1}^5$.

The total number of favorable outcomes is the sum of the two cases: $4 \times \binom{4}{2} \times \binom{4}{1}^3 + (13 - 4) \times \binom{4}{1}^5$

(b) There are 3 cases: no hearts and no spades, one heart and one spade and two hearts and two spades.

- Case I: no hearts and no spades. So we must pick 5 cards from the clubs and diamonds: $\binom{26}{5}$ ways.
- Case II: one heart and one spade. Choose one heart and one spade: $\binom{13}{1}^2$ ways. Next choose 3 cards from the other 2 suites: $\binom{26}{3}$ ways. So $\binom{13}{1}^2 \times \binom{26}{3}$ outcomes.

- Case III: two hearts and two spades. Choose two hearts and two spades: $\binom{13}{2}^2$. Next choose 1 card from the other suites: $\binom{26}{1}$. So $\binom{13}{2}^2 \times \binom{26}{1}$ outcomes.

So the number of favorable outcomes is $\binom{26}{5} + \binom{13}{1}^2 \times \binom{26}{3} + \binom{13}{2}^2 \times \binom{26}{1}$.

Problem 54. There are 4 cases: no vowels, one vowel, two vowels and three vowels.

- Case I: no vowels. So we want an 8-permutation of the remaining 21 consonants: $P(21, 8)$
- Case II: one vowel. Choose the vowel: 5 ways. Next decide where it goes: 8 possible slots. For the remaining 7 slots, we want a 7-permutation of the 21 consonants: $P(21, 7)$ ways. So the total is $5 \times 8 \times P(21, 7)$.
- Case III: two vowels. Choose vowels: $\binom{5}{2}$ ways. Next decide where they go: since they have to be in alphabetical order, we just need to choose the two slots for the two vowels: $\binom{8}{2}$ ways. Now for the remaining slots, we want a 6-permutation of the 21 consonants: $P(21, 6)$ ways. So a total of $\binom{5}{2} \times \binom{8}{2} \times P(21, 6)$.
- Case IV: three vowels. Choose vowels: $\binom{5}{3}$ ways. Decide where they go: again, just need to choose three slots since the order is determined: $\binom{8}{3}$ ways. Next we want a 5-permutation of the 21 consonants: $P(21, 5)$. So total is: $\binom{5}{3} \times \binom{8}{3} \times P(21, 5)$.

So the final answer is $P(21, 8) + 5 \times 8 \times P(21, 7) + \binom{5}{2} \times \binom{8}{2} \times P(21, 6) + \binom{5}{3} \times \binom{8}{3} \times P(21, 5)$.

Problem 62. There are many different ways to do this problem. Here is one. First write down the two 6s:

$$6 \quad 6$$

Now imagine distributing the three 3s among these 6s. Since no three comes before the first 6, there are 4 ways of doing this: all three 3s go in between the 6s, or only two 3s go in between, or only one 3 goes in between or no 3s go in between. Once the 3s have been arranged, arrange the 8 and 2 among the 6s and 3s. We can think of the places where they can go as slots:

$$slot1 \quad 6 \quad slot2 \quad 3 \quad slot3 \quad 6 \quad slot4 \quad 3 \quad slot5 \quad 3 \quad slot6$$

So there are 6 slots to put the 8. Once the 8 is placed, there will be 7 slots for the 2. So the answer is $4 \times 6 \times 7$.

Problem 66. This is the same as picking m + n people out of the 100. Any choice of m + n people out of 100 can be separated into the bottom m and the top n. This gives a group of n people and a group of m other people whose heights are all less than the heights of the people in the first group. On the other hand, given two such groups, their union is a group of m + n people. So the answer is $\binom{100}{m+n}$.

Problem 67. Since the three values are all different, think of the outcomes as tuples (L, M, H) where L represents the smallest value choosen, M the middle value and H the highest value.

(a) We want $L + M + H$ to be an even number. There are two ways the sum of three numbers can be even: either all three numbers are even or two are odd and one is even. So we break into two cases:

- Case I: all three numbers are even. We choose 3 numbers from the 45 even ones: $\binom{45}{3}$ ways.
- Case II: two are odd and one is even: $\binom{45}{2} \times \binom{45}{1}$ ways.

So the total is $\binom{45}{3} + \binom{45}{2} \times \binom{45}{1}$

(b) This is very much like (a) except now instead of even and odd, we need to look at what remainder a number leaves when divided by 3. This could be 0, 1 or 2. And there are four cases where $L + M + H$ will be divisible by 3: all are 0 mod 3, or all are 1 mod 3, or all are 2 mod 3, or there is one of each. Each case is easy to handle, just like in part(a).

(b) Again we look at what remainders a number can leave when divided by 4. It can be 0, 1, 2 or 3. Now there are 5 cases to handle!!

Problem 72. This problem is quite tricky and really uses the material in section 5.3. First arrange the five 0s. Since the 0s are identical there is just one way to do this.

$$0 \quad 0 \quad 0 \quad 0 \quad 0$$

Now we want to distribute the ten 1s among these 0s. We think of all the places to put 1s as being boxes:

$$\textit{box } 1 \quad 0 \quad \textit{box } 2 \quad 0 \quad \textit{box } 3 \quad 0 \quad \textit{box } 4 \quad 0 \quad \textit{box } 5 \quad 0 \quad \textit{box } 6$$

So we want to distribute ten identical 1s among six distinct boxes. But we have be sure that boxes 2, 3, 4 and 5 contain atleast one 1 each, so that we avoid consecutive 0s. We handle this requirement by first throwing a 1 into each of boxes 2, 3, 4 and 5. We are left with six 1s, which we want to distribute into six boxes. There are $\binom{6+6-1}{6}$ ways to do this.

2 Section 5.3

Problem 4. Suppose the friends are A, B and C. We think of the six nights as six slots:

__ __ __ __ __ __

We want to place one of A, B or C in each slot, making sure that no one is placed in more than 3 slots. Since there are repetitions, we need to use the formula for arrangements with repetitions. But for that we need to know exactly how many As, Bs and Cs we have. So there are 3 cases: each friend is invited twice, or one friend is invited thrice, one of them twice and one of them just once, or two friends are invited thrice each and the other one not at all.

- Case I: each friend is invited twice. So we have 2 As, 2 Bs and 2 Cs to place in six slots. There are $\frac{6!}{2!2!2!}$ ways to do this.
- Case II: one friend is invited thrice, one twice and one once. First we need to decide which friend gets invited thrice, which one twice and which one once. There are 3! ways to do this. Once this has been done we need to arrange them in the 6 slots. We have 3 of one letter, 2 of one letter and 1 of the third (for example we may have 3 As, 2 Bs and 1C). So there are $\frac{6!}{3!2!1!}$ ways to do this. So the total is $3! \times \frac{6!}{3!2!1!}$
- Case III: two friends are invited thrice each. First decide who these two friends are: $\binom{3}{2}$ ways. Now arrange them. We have 3 of one letter and 3 of another letter (for exmaple, we may have 3 As and 3Cs). There are $\frac{6!}{3!3!}$ ways to do this. So the total is $\binom{3}{2} \times \frac{6!}{3!3!}$

The final answer is $\frac{6!}{2!2!2!} + 3! \times \frac{6!}{3!2!1!} + \binom{3}{2} \times \frac{6!}{3!3!}$

Problem10. (a) First distribute the apples, then the pears. For the apples we want to distribute 5 identical objects into 3 distinct boxes: $\binom{5+3-1}{5}$ ways. For the pears, we need to make sure each person gets a pear. So first give each one a pear. Now we have 3 identical pears left, which we want to distribute into 3 distinct boxes: $\binom{3+3-1}{3}$. So the total is $\binom{5+3-1}{5} \times \binom{3+3-1}{3}$.

(b) First distribute the apples. Since the apples are distinct, we think of them as 5 slots:

$$\underline{}_{\,apple\ 1}\quad \underline{}_{\,apple\ 2}\quad \underline{}_{\,apple\ 3}\quad \underline{}_{\,apple\ 4}\quad \underline{}_{\,apple\ 5}$$

Suppose the 3 people are A, B and C. We want to place one letter in each of the slot without any restrictions. So there are 3^5 ways to do this. Now we come to the pears. We think of them as slots too:

$$\underline{}_{\,pear\ 1}\quad \underline{}_{\,pear\ 2}\quad \underline{}_{\,pear\ 3}\quad \underline{}_{\,pear\ 4}\quad \underline{}_{\,pear\ 5}\quad \underline{}_{\,pear\ 6}$$

Again we need to arrange the 3 letters in the slots, but we need to make sure that each letter occurs atleast once. It is best to count the complement: the arrangements where some letter does not occur at all. First let us count all arrangements without any restrictions. This is like the apple case. So there are 3^6 ways to do this. Now for the complement, there are 2 cases: only two letters occur or only one letter occurs.

- Case I: two letters occur. First choose the two letters: $\binom{3}{2}$ ways. Now arrange them without restriction: 2^6 ways. But two of these are cases where only one of the 2 letters gets used. So subtract them: $2^6 - 2$. So a total of $\binom{3}{2} \times (2^6 - 2)$ ways for this case.
- Case II: only one letter occurs. Choose the letter: $\binom{3}{1}$ ways. Once the letter is decided there is just one way to arrange it in the slots. So $\binom{3}{1}$ ways for this case.

So the number of ways to distribute pears is $3^6 - \left[\binom{3}{2} \times (2^6 - 2) + \binom{3}{1}\right]$. The final answer is: $3^5 \times \left[3^6 - \left[\binom{3}{2} \times (2^6 - 2) + \binom{3}{1}\right]\right]$.