HW11 Solutions

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1 Appendix A.4

Problem 8. Let us arrange the numbers in strictly increasing order (we can do this since they are all distinct). So we have $1 \le k_1 < k_2 < \cdots < k_{n+1} \le 2n$. Now if the difference between k_{i+1} and k_i is always at least 2, then the difference between k_1 and k_{n+1} is at least $2n$. But the difference between two numbers laying between 1 and $2n$ is at most $2n-1$. So the difference between some k_i and k_{i+1} must be 1, which means they are consecutive.

Problem 11. For each number from 1 to 14 consider the highest power of 2 dividing it. The part that is left over must be an odd number between 1 and 14, of which there are only 7. That is, suppose that the 8 numbers given to us are $n_1, \ldots, n_8$. For each i between 1 and 8 write $n_i = 2^{m_i}k_i$, where 2^{m_i} is the highest power of 2 dividing n_i, and k_i is an odd number. Notice that k_i is between 1 and 14. There are only 7 odd numbers between 1 and 14. So there are $i \neq j$ such that $k_i = k_j$. Now the one with the smaller m divides the other one. That is, if $m_i \le m_j$, then n_i divides n_j, and if $m_j \le m_i$, then n_j divides n_i.

Problem 12. Let us look at our n integers modulo $n-1$. There are $n-1$ congruence classes mod $n-1$. Since we have n integers, there must be i and j such that $i \equiv j \mod n-1$. But then $i - j \equiv 0 \mod n-1$, meaning that it is divisible by $n-1$.

Problem 17. The maximum possible sum that a subset of S can have is $51 + 52 + \cdots + 60 = 555$. But there are $2^{10} - 1 = 1023$ non–empty subsets of S. So there must be non–empty S_1 and S_2, *different* subsets of S, with the same sum. However, S_1 and S_2 may not be disjoint. But notice that we cannot have $S_1 \subset S_2$. This is because S_1 and S_2 have the same sum. If $S_1 \subset S_2$, then since S_1 and S_2 are different sets, there would be a number between 1 and 60 that is in S_2 but not in S_1, which would make the sum of S_2 strictly greater than the sum of S_1. Similarly, $S_2 \not\subset S_1$. Therefore, $S_1 \cap S_2 \neq S_1$ and $S_1 \cap S_2 \neq S_2$. So $S'_1 = S_1 \setminus (S_1 \cap S_2)$ and $S'_2 = S_2 \setminus (S_1 \cap S_2)$ are disjoint non–empty subsets of S. Moreover, $\sum S_1 = \sum S'_1 + \sum S_1 \cap S_2$, and $\sum S_2 = \sum S'_2 + \sum S_1 \cap S_2$. Since $\sum S_1 = \sum S_2$, it follows that $\sum S'_1 = \sum S'_2$.

Problem 19. For each $1 \le i \le 49$, let a_i be the number of hours studied on day i. We are told that $1 \le a_i$. For each $1 \le i \le 49$, put $S_i = a_1 + a_2 + \cdots + a_i$. Thus we have

$$1 \le S_1 < S_2 < \cdots < S_i < \cdots < S_{49} \le 77$$

Now, let us add 20 to each member of this sequence. Notice that since $S_i < S_{i+1}$, we will have $S_i + 20 < S_{i+1} + 20$. So we get the sequence

$$1 \le S_1 + 20 < S_2 + 20 < \cdots < S_i + 20 < \cdots < S_{49} + 20 \le 97$$

Thus we have 98 numbers in the set $\{S_1 < S_2 < \cdots < S_{49}\} \cup \{S_1 + 20 < S_2 + 20 < \cdots < S_{49} + 20\}$ that are all between 1 and 97. Therefore, two of them must be equal. However, the S_i are all different from each other and the $S_i + 20$ are all different from each other. It follows that there must $i < j$ such that $S_j = S_i + 20$ and so $S_j - S_i = 20$. So we have

$$S_j = a_1 + a_2 + \cdots + a_i + a_{i+1} + \cdots + a_j$$
$$S_i = a_1 + a_2 + \cdots + a_i$$
$$S_j - S_i = a_{i+1} + \cdots + a_j = 20.$$

So in the period of consecutive days from day $i + 1$ to day j, the student studies for a total of exactly 20 hours.

Problem 21. This problem is very tricky. Let the sequence of $n^2 + 1$ numbers be $x_1, x_2, \ldots, x_{n^2+1}$. Suppose for a contradiction that there are no increasing or decreasing subsequences of length $n + 1$. For each $1 \le i \le n^2 + 1$, let a_i be the length of any longest increasing subsequence starting at x_i. Notice that $1 \le a_i$, and by our assumption, $a_i \le n$. Let b_i be the length of any longest decreasing subsequence starting at x_i. Again, $1 \le b_i \le n$. Thus are are only n^2 pairs of numbers (a_i, b_i) for $1 \le i \le n^2 + 1$. Therefore, there is $i < j$ such that $a_i = a_j$ *and* $b_i = b_j$. We will get a contradiction from this. Since the x_i are distinct, either $x_i < x_j$ or $x_j < x_i$. Suppose that $x_i < x_j$. There is an increasing subsequence of length $a_j = a_i$ starting at x_j. But we can add x_i at the beginning of this subsequence to get an increasing subsequence of length $a_i + 1$ starting at x_i (because $x_i < x_j$). But this is a contradiction because a_i was supposed to the length of any longest increasing subsequence starting at x_i. Similarly, if $x_i > x_j$, we can get a decreasing subsequence of length $b_i + 1$ starting at x_i, which is again a contradiction.