HW10 Solutions

Dilip Raghavan

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1 Section 8.2

Problem 12. Let $\mathcal{U}$ be the set of all 4–digit numbers, including those with leading 0s. For each i between 0 and 9, let A_i be the set of all 4–digit numbers in which the digit i occurs exactly twice. Now, $|\mathcal{U}| = 10^4$. For each i, $|A_i| = \binom{4}{2} \times 9^2$ – first choose the 2 slots for the exactly 2 occurances of i, and then arrange any of the other 9 digits in the two remaining slots. Therefore, $S_1 = \binom{10}{1} \times \binom{4}{2} \times 9^2$. Now, given a pair $i \neq j$, $\left|A_i \cap A_j\right| = \binom{4}{2}$ – choose 2 slots for the two occurrences of i, and then j must go into the remaining 2 slots. Therefore, $S_2 = \binom{10}{2} \times \binom{4}{2}$. Notice that all triple intersections are empty because a 4–digit number cannot have 3 digits ocurring exactly twice. Thus, $\left|\bar{A}_0 \cap \cdots \cap \bar{A}_9\right| = |\mathcal{U}| - S_1 + S_2 = 10^4 - \binom{10}{1} \times \binom{4}{2} \times 9^2 + \binom{10}{2} \times \binom{4}{2}$.

Problem 18. For a real number x, let $\lceil x \rceil$ denote the least integer *greater than or equal* to x. Note that the number of odd numbers in the set $\{1, \ldots, n\}$ is equal to $\lceil \frac{n}{2} \rceil$. Now, let $\mathcal{U}$ be the set of all possible arrangements of $1, 2, \ldots, n$. For each odd number i between 1 and n, let A_i be the set of arrangements with i ocurring in slot i (so i is *not* deranged). $|\mathcal{U}| = n!$. For each odd number i, $|A_i| = (n-1)!$. So $S_1 = \binom{\lceil \frac{n}{2} \rceil}{1}(n-1)!$. For each pair $i \neq j$ of odd numbers, $\left|A_i \cap A_j\right| = (n-2)!$, and so $S_2 = \binom{\lceil \frac{n}{2} \rceil}{2}(n-2)!$. Similarly, for each k between 1 and $\lceil \frac{n}{2} \rceil$, $S_k = \binom{\lceil \frac{n}{2} \rceil}{k}(n-k)!$. Since $|\mathcal{U}| = n!$ can be written as $(-1)^0\binom{\lceil \frac{n}{2} \rceil}{0}(n-0)!$, the answer is

$$\sum_{k=0}^{\lceil \frac{n}{2} \rceil} (-1)^k \binom{\lceil \frac{n}{2} \rceil}{k}(n-k)!$$

Problem 30. We have 9 slots in which to arrange the 3 as, 3 bs and 3 cs. Let $|\mathcal{U}|$ be the set of all possible arrangements of these letters in the 9 slots. Let A_1 be the set of arrangements with consecutive as. Let A_2 be the set of arrangements with consecutive bs and let A_3 be the set of arrangements with consecutive cs. We want to count $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3\right|$. However, counting the sizes of the $|A_i|$, the $\left|A_i \cap A_j\right|$, and the $\left|A_i \cap A_j \cap A_k\right|$ is a bit tricky and each requires a kind of inclusion–exclusion. First of all, $|\mathcal{U}| = \frac{9!}{(3!)^3}$. To count $|A_1|$, we can glue together 2 as. Now, there are $\frac{8!}{(3!)^2}$ ways to arrange these symbols. But this double counts the arrangements with all three as consecutive. So we must subtract these: $\frac{7!}{3!^2}$. So $|A_1| = \frac{8!}{(3!)^2} - \frac{7!}{3!^2}$. The same for $|A_2|$ and $|A_3|$. So $S_1 = 3 \times \left(\frac{8!}{(3!)^2} - \frac{7!}{3!^2}\right)$. Now to count $|A_1 \cap A_2|$, we can glue together 2 as and 2 bs. There are $\frac{7!}{3!}$ ways to arrange these symbols. But again we must subtract the arrangements with all 3 as consecutive, and also the arrangements with all 3 bs consecutive. So $\frac{7!}{3!} - 2 \times \frac{6!}{3!}$. But now we have subtracted too much because we have subtracted the arrangements with *both* 3 consecutive as *and* 3 consecutive bs twice. So we must add them back. So we get $|A_1 \cap A_2| = \frac{7!}{3!} - 2 \times \frac{6!}{3!} + \frac{5!}{3!}$. The same for $|A_2 \cap A_3|$ and also $|A_3 \cap A_1|$. So $S_2 = 3 \times \left(\frac{7!}{3!} - 2 \times \frac{6!}{3!} + \frac{5!}{3!}\right)$. Using a similar inclusion–exclusion type argument, $S_3 = |A_1 \cap A_2 \cap A_3| = 6! - 3 \times 5! + 3 \times 4! - 3!$. So the answer is $\frac{9!}{(3!)^3} - 3 \times \left(\frac{8!}{(3!)^2} - \frac{7!}{3!^2}\right) + 3 \times \left(\frac{7!}{3!} - 2 \times \frac{6!}{3!} + \frac{5!}{3!}\right) - (6! - 3 \times 5! + 3 \times 4! - 3!)$.

Problem 33. A design may be thought of as an arrangement of 3 out of the 6 possible colors (the first slot represents the top piece, the second slot the middle piece, and the third slot the bottom piece). Thus there are a total of $P(6,3)$ possible designs. Now, let $\mathcal{U}$ be the set of all possible selections of 8 designs. So $|\mathcal{U}| = \binom{P(6,3)}{8}$. Now for each i between 1 and 6, let A_i be the set of selections of 8 designs *not* using color i. So we want $\left|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_6\right|$. For each i, $|A_i| = \binom{P(5,3)}{8}$, and so $S_1 = \binom{6}{1} \times \binom{P(5,3)}{8}$. Similarly, $S_2 = \binom{6}{2} \times \binom{P(4,3)}{8}$. S_3 and above are equal to 0 because, there are only 6 possible designs using 3 colors. So the answer is $|\mathcal{U}| - S_1 + S_2 = \binom{P(6,3)}{8} - \binom{6}{1} \times \binom{P(5,3)}{8} + \binom{6}{2} \times \binom{P(4,3)}{8}$

Problem 34. Label the people 1 to 6 going clockwise (or counter clockwise; it doesn't matter). Now, for each i between 1 to 5, i cannot get the same entree as $i+1$, and 6 cannot get the same entree as 1. Let $\mathcal{U}$ be the set of all possible ways to distribute the entrees. For each i between 1 and 5, let A_i be the set of distributions with i getting the same entree as $i+1$, and let A_6 be the set of distributions with 6 getting the same entree as 1. Now, $|\mathcal{U}| = m^6$ (m choices for each of the six people). For each i, $|A_i| = m^5 - m$ choices for the two people getting the same entree (either i and $i+1$, or 6 and 1 depending on what i is), and m choices each for the remaining 4 people. So $S_1 = \binom{6}{1} \times m^5$. By similar reasoning, $S_2 = \binom{6}{2} \times m^4$, $S_3 = \binom{6}{3} \times m^3$, $S_4 = \binom{6}{4} \times m^2$, $S_5 = \binom{6}{5} \times m$ (because if an intersection of 5 of the A_i is taken, then everyone must end up getting the same entree), and $S_6 = \binom{6}{6} \times m$ (again everyone gets the same entree). So the answer is $m^6 - \binom{6}{1} \times m^5 + \binom{6}{2} \times m^4 - \binom{6}{3} \times m^3 + \binom{6}{4} \times m^2 - \binom{6}{5} \times m + \binom{6}{6} \times m$.

Problem 39. This problem really requires theorem 2 (page 329), which we did not cover.