HW1 Solutions

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1 Section 5.1

Problem 16. We model the outcomes as 4–tuples $(R1, W1, R2, W2)$, where $R1$ represents the first roll of the red die and $R2$ the second, and similarly for $W1$ and $W2$.

a. There are 6 possible outcomes for each of $R1$, $R2$, $W1$ and $W2$. So there are a total of 6^4 outcomes.

b. We need R1 = R2 and W1 = W2. So if R1 and W1 are choosen, R2 and W2 are determined. There are 6 possible outcomes for each of R1 and W1. So there are 6^2 favorable outcomes. So the probability is

$$\frac{\text{no. of favorable outcomes}}{\text{no. of total outcomes}} = \frac{6^2}{6^4}.$$

c. The sum can be anything from 2 to 12. It is best to break up the problem into cases depending on what the sum is. Notice that given a fixed value for the sum, if the value of $R1$ is choosen, then the value of $W1$ is automatically determined. So for any given sum, it is enough to figure out how many outcomes there are for $R1$ and for $R2$.

   (a) The sum is 2. $R1$ can only be 1. Same for $R2$. So 1^2 outcomes.
   (b) The sum is 3. $R1$ can be either 1 or 2. Same for $R2$. So 2^2 outcomes.
   (c) The sum is 4. $R1$ can be anything from 1 to 3. Same for $R2$. So 3^2 outcomes.
   (d) The sum is 5. $R1$ can be anything from 1 to 4. Same for $R2$. So 4^2 outcomes.
   (e) The sum is 6. $R1$ can be anything from 1 to 5. Same for $R2$. So 5^2 outcomes.
   (f) The sum is 7. $R1$ can be anything from 1 to 6. Same for $R2$. So 6^2 outcomes.
   (g) The sum is 8. $R1$ can be anything from 2 to 6. Same for $R2$. So 5^2 outcomes.
   (h) The sum is 9. $R1$ can be anything from 3 to 6. Same for $R2$. So 4^2 outcomes.
   (i) The sum is 10. $R1$ can be anything from 4 to 6. Same for $R2$. So 3^2 outcomes.
   (j) The sum is 11. $R1$ can be anything from 5 to 6. Same for $R2$. So 2^2 outcomes.
   (k) The sum is 12. $R1$ has to be 6. Same for $R2$. So 1^2 outcomes.

   So the number of favorable outcomes is:

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2.$$

d. The number of outcome where $R1 + W1 \neq R2 + W2$ can be gotten by subtracting the answer to part c. from the total number of outcomes. Since the situation is symmetrical, the number of outcomes where $R1 + W1 < R2 + W2$ is half of the number of outcomes where $R1 + W1 \neq R2 + W2$. So the number of favorable outcomes is

$$\frac{1}{2}\left[6^4 - \left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2\right]\right].$$

Problem 22. The first slot can have any of 0, 1 or 2. So there are 3 possibilities for the first slot. Every slot after the first one can have anything *except* what is in the slot right before it. So there are 2 possibilities for every slot after the first one. So there are a total of 3×2^9 outcomes.

Problem 23. In all three problems it is best to first count the possibilities between 1,000 and 9,999 and then to deal with 10,000 separately.

a. For the numbers between 1,000 and 9,999, the first slot can be anything between 1 and 9, so 9 possibilities. The second slot can be anything from 0 to 9 *except* what is in the first slot. So again 9 possibilities. The third one can be anything from 0 to 9 *except* what is in the first two slots, so 8 possibilities. The fourth one can be anything *except* what is in the first three, so 7 possibilities. So there are $9 \times 9 \times 8 \times 7$ such numbers between 1,000 and 9,999. 10,000 does not have distinct digits, so the answer is $9 \times 9 \times 8 \times 7$.

b. Between 1,000 and 9,999, the first slot can be anything from 1 to 9 *except* 2 and 4, so 7 possibilities. The remaining slots can be anything from 0 to 9 *except* 2 and 4, so 8 possibilities. So there are 7×8^3 such numbers between 1,000 and 9,999. 10,000 satisfies the condition of not having 2 or 4. So the answer is $7 \times 8^3 + 1$

c. First note that 10,000 is excluded because it does not have distinct digits. For the numbers between 1,000 and 9,999, count the complement – the number of numbers between 1,000 and 9,999 that have distinct digits and have *neither* 2 *nor* 4. The first slot can be anything between 1 and 9 except 2 and 4, so 7 possibilities. The second slot can be anything between 0 and 9 except 2, 4 and whatever is in the first slot; so 7 possibilities. The third one can be anything between 0 and 9 except 2, 4 and whatever is in the first two slots; so 6 possibilities. The fourth slot can be anything between 0 and 9 except 2, 4 and whatever is in the first 3 slots; so 5 possibilities. So the total is $7 \times 7 \times 6 \times 5$. This is the complement. From part a., there are $9 \times 9 \times 8 \times 7$ numbers with distinct digits. So the number with distinct digits and *either* 2 *or* 4 is $9 \times 9 \times 8 \times 7 - 7 \times 7 \times 6 \times 5$.

Problem 26. The total number of outcomes is the number of numbers between 1 and 9,999, which is 9,999. To count the favorable outcomes first decide where the 8 and the 9 will go. 8 can go into any of the 4 slots, so 4 possibilities. Once the 8 is fixed, 9 can go into any of the remaining 3 slots. So there are 4×3 ways to arrange the 8 and the 9. Now the remaining two slots can have any digit between 0 and 7. So there are 8^2 ways to fill in the remaining two slots. So there are a total of $4 \times 3 \times 8^2$ favorable outcomes.

Problem 27. Count the complement – i.e. the number of sequences with BAD. First figure out how many ways there are to place BAD. The leading B can go into any of the first three slots; but not into the fourth or fifth ones because otherwise there will not be enough room for the A and the D.So there are 3 ways to arrange BAD. Once this is done there are 2 slots left over. Any of the four letters can go into those. So there are a total of 3×4^2 sequences with BAD.

The total number of sequences is 4^5 because there are 4 possibilities for each of the 5 slots. So the number of sequences without BAD is $4^5 - 3 \times 4^2$.

Problem 35. We model the outcomes as pairs (F, S), where F represents the first card choosen and S represents the second card. The first card can be any of 50. So there are 50 outcomes for F. The second card can be anything other than the first, so 49 possibilities. So there are 49×50 total outcomes.

To count the number of favorable outcomes, break the problem into two cases: $F = 2S$ or $S = 2F$. For $F = 2S$, S can be anything from 1 to 25, and once S is choosen, F is determined. So there are 25 outcomes. For the second case, when $S = 2F$, again there are 25 outcomes because F can be anything from 1 to 25. So there are $25 + 25$ favorable outcomes.

Problem 37. Each die has 6 possible values. So there are 6^3 total outcomes.

Counting the number of favorable outcomes is a bit tricky and there are many ways to do it. We present one approach here. To count the favorable outcomes, we first look at the 3–tuples (L, M, H), where L represents the lowest value, M represents the middle value and H represents the highest value. Now, we know that $H = 2L$. The middle value M could be strictly between H and L or it could be equal to L or to H. We break the problem into these three cases.

- Case I: $L < M < H = 2L$. There are 3 possible values for the 3–tuple (L, M, H): (2, 3, 4), (3, 4, 6) and (3, 5, 6). Once the tuple has been fixed, we must figure out how many ways there are to assign the 3 values to the 3 dice. L could be assigned to any of the 3 dice, so 3 possibilities. M can be assigned to any of the remaining 2, and H to the last one that remains. So there are $3 \times 2 \times 1$ ways to assign the values for a fixed tuple. So there are a total of $3 \times 3 \times 2 \times 1$ outcomes for this case.

- Case II: $L = M < H = 2L$. There are 3 possible values for the 3–tuple (L, M, H): (1, 1, 2), (2, 2, 4) and (3, 3, 6). Again, once the tuple is fixed we must count how many ways there are to assign the values to the dice. Given a tuple, we just need to decide which which dice gets to highest value, since the other two values are same. So there are 3 ways to assign the values to the dice per tuple. So there are 3×3 outcomes for this case.

- Case III: $L < M = H = 2L$. This case is just like case II. Running through the same reasoning, there are 3×3 outcomes for this case too.

So there are in total $3 \times 3 \times 2 \times 1 + 3 \times 3 + 3 \times 3 = 36$ favorable outcomes.

2 Section 5.2

Problem 16. The total number of hands is $\binom{52}{5}$

(a) If there are four Aces, we just need to decide the fifth card. It could be any of the remaining $52-4 = 48$ cards. So there are 48 favorable outcomes.

(b) You must first decide the kind. Once the kind is fixed, the problem is just like part (a); there are 48 outcomes per kind. Since there are 13 kinds, there are 13×48 favorable outcomes.

(c) We must first choose the two ranks for the 2 pairs. There are $\binom{13}{2}$ ways to do this. Then we must choose a pair from each of those ranks. There are $\binom{4}{2}^2$ ways to do this. Finally, the fifth card can be any card of a rank other than the two ranks we have chosen; so 44 possibilities. In total, we have $\binom{13}{2} \times \binom{4}{2}^2 \times 44$ favorable outcomes.

(d) First decide the two ranks; there are $\binom{13}{2}$ ways for this. Then decide which of the two will have 3 of a kind; 2 ways to do this. Now we must choose 3 cards from one and a pair from the other; there are $\binom{4}{3} \times \binom{4}{2}$ ways to do this. In total, there are $\binom{13}{2} \times 2 \times \binom{4}{3} \times \binom{4}{2}$ favorable outcomes.

(e) First decide the starting value of the sequence. Since the Ace can be both a low value and a high value, there are 10 starting places. Then for each of the 5 values we must give a suite. There are 4^5 ways to do this. So there are $4^5 \times 10$ favorable outcomes.

(f) No pairs means that the five cards must all be of different ranks. So first choose a set of 5 ranks. There are $\binom{13}{5}$ ways to do this. Now decorate each choosen rank with a suite. There are 4^5 ways to do this. So $4^5 \times \binom{13}{5}$ favorable outcomes.

Problem 17. (a) Consider the possible values of the pair (M, W), where M represents the number of men and W the number of women. There are four possible values for this pair: $(2,4), (2,5), (2,6), (3,6)$. We break the problem into 4 cases:

- Case $(2,4)$: $\binom{4}{2} \times \binom{6}{4}$ ways
- Case $(2,5)$: $\binom{4}{2} \times \binom{6}{5}$ ways
- Case $(2,6)$: $\binom{4}{2} \times \binom{6}{6}$ ways
- Case $(3,6)$: $\binom{4}{3} \times \binom{6}{6}$ ways

In total, there are $\binom{4}{2} \times \binom{6}{4} + \binom{4}{2} \times \binom{6}{5} + \binom{4}{2} \times \binom{6}{6} + \binom{4}{3} \times \binom{6}{6}$ ways.

(b) 3 cases depending on whether there are 3, 4 or 5 people.

- Exactly 3 people: $\binom{9}{3}$ ways
- Exactly 4 people: $\binom{9}{4}$ ways
- Exactly 5 people: $\binom{9}{5}$ ways

In total, $\binom{9}{3} + \binom{9}{4} + \binom{9}{5}$ ways.

(c) Count the complement. Total number of 5 person committees is $\binom{10}{5}$. If all 3 O'Hara sisters are there, we need to select 2 people out of 7; so $\binom{7}{2}$ committees with all three sisters. So $\binom{10}{5} - \binom{7}{2}$ such committees.

(d) There are many ways to break up into cases. Here is one:

- Case I: There are exactly 2 women. Total number of such committees is $\binom{6}{2} \times \binom{4}{2}$. If both Mr. and Mrs. Biggins are there, we have $\binom{3}{1} \times \binom{5}{1}$ possibilities for the remaining 2 people. So there are $\binom{6}{2} \times \binom{4}{2} - \binom{3}{1} \times \binom{5}{1}$ such committees without both Mr. and Mrs. Biggins.

- Case II: There are exactly 3 women. Total number is $\binom{6}{3} \times \binom{4}{1}$. If both Mr. and Mrs. Biggins are present, we have $\binom{5}{2}$ possibilities for the remaining 2 women. So there are $\binom{6}{3} \times \binom{4}{1} - \binom{5}{2}$ such committees without them both.

- Case III: There are 4 women. Then Mr. Biggins cannot be there. So there are $\binom{6}{4}$ possibilities.

In all, there are $\left[\binom{6}{2} \times \binom{4}{2} - \binom{3}{1} \times \binom{5}{1}\right] + \left[\binom{6}{3} \times \binom{4}{1} - \binom{5}{2}\right] + \binom{6}{4}$ committees.

Problem 22. Let's first count the total number of outcomes. Imagine the 13 players lined up in a row. We can give the first player $\binom{52}{4}$ hands. $\binom{52-4}{4}$ hands to the second player and so on. So there are

$$\binom{52}{4} \times \binom{52-4}{4} \times \binom{52-4\times 2}{4} \cdots \binom{52-4\times 12}{4} = \prod_{i=0}^{12} \binom{52-4i}{4}$$

total outcomes.

To get the number of favorable outcomes, we take one suite at a time and deal the cards in that suite to the 13 players, with each player getting one card. Given a suite there are 13! ways to deal the 13 cards in that suite to the 13 players. We do this for all 4 four suites, one by one. So there are $(13!)^4$ favorable outcomes.