KdV PRESERVES WHITE NOISE

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ABSTRACT. It is shown that white noise is an invariant measure for the Korteweg-deVries equation on $\mathbb{T}$. This is a consequence of recent results of Kappeler and Topalov establishing the well-posedness of the equation on appropriate negative Sobolev spaces, together with a result of Cambronero and McKean that white noise is the image under the Miura transform (Ricatti map) of the (weighted) Gibbs measure for the modified KdV equation, proven to be invariant for that equation by Bourgain.

1. KdV on $H^{-1}(\mathbb{T})$ and White Noise

The Korteweg-deVries equation (KdV) on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$,

$$u_t - 6uu_x + u_{xxx} = 0, \qquad u(0) = f \tag{1.1}$$

defines nonlinear evolution operators

$$\mathcal{S}_t f = u(t) \tag{1.2}$$

$-\infty < t < \infty$ on smooth functions $f : \mathbb{T} \to \mathbb{R}$.

Theorem 1.1. *(Kappeler and Topalov* [KT1]*) $\mathcal{S}_t$ extends to a continuous group of nonlinear evolution operators*

$$\bar{\mathcal{S}}_t : H^{-1}(\mathbb{T}) \to H^{-1}(\mathbb{T}). \tag{1.3}$$

In concrete terms, take $f \in H^{-1}(\mathbb{T})$ and let f_N be smooth functions on $\mathbb{T}$ with $\|f_N - f\|_{H_{-1}(\mathbb{T})} \to 0$ as $N \to \infty$. Let $u_N(t)$ be the (smooth) solutions of (1.1) with initial data f_N. Then there is a unique $u(t) \in H^{-1}(\mathbb{T})$ which we call $u(t) = \bar{\mathcal{S}}_t f$ with $\|u_N(t) - u(t)\|_{H^{-1}(\mathbb{T})} \to 0$.

White noise on $\mathbb{T}$ is the unique probability measure Q on the space $\mathcal{D}(\mathbb{T})$ of distributions on $\mathbb{T}$ satisfying

$$\int e^{i\langle \lambda, u\rangle} dQ(u) = e^{-\frac{1}{2}\|\lambda\|_2^2} \tag{1.4}$$

for any smooth function λ on $\mathbb{T}$ where $\|\cdot\|_2^2 = \langle \cdot, \cdot \rangle$ are the $L^2(\mathbb{T}, dx)$ norm and inner product (see [H]).

Let $\{e_n\}_{n=0,1,2,\ldots}$ be an orthonormal basis of smooth functions in $L^2(\mathbb{T})$ with $e_0 = 1$. White noise is represented as $u = \sum_{n=0}^{\infty} x_n e_n$ where and x_n are independent Gaussian random variables, each with mean 0 and variance 1. Hence Q is supported in $H^{-\alpha}(\mathbb{T})$ for any $\alpha > 1/2$.

Mean zero white noise Q_0 on $\mathbb{T}$ is the probability measure on distributions u with $\int_{\mathbb{T}} u = 0$ satisfying

$$\int e^{i\langle \lambda, u\rangle} dQ_0(u) = e^{-\frac{1}{2}\|\lambda\|_2^2} \tag{1.5}$$

for any mean zero smooth function λ on $\mathbb{T}$. It is represented as $u = \sum_{n=1}^{\infty} x_n e_n$.

Recall that if $f : X_1 \to X_2$ is a measurable map between metric spaces and Q is a probability measure on $(X_1, \mathcal{B}(X_1))$, then the *pushforward* f^*Q is the measure on X_2 given by $f^*Q(A) = Q(\{x \ : \ f(x) \in A\})$ for any Borel set $A \in \mathcal{B}(X_2)$.

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Our main result is:

Theorem 1.2. *White noise Q_0 is invariant under KdV; for any $t \in \mathbb{R}$,*

$$\bar{\mathcal{S}}_t^* Q_0 = Q_0. \tag{1.6}$$

Remarks. 1. In terms of classical solutions of KdV, the meaning of Theorem 1.2 is as follows. Let f_N, $N = 1, 2, \ldots$ be a sequence of smooth mean zero random initial data approximating mean zero white noise. For example, one could take $f_N(\omega) = \sum_{n=1}^N x_n(\omega) e_n$ where x_n and e_n are as above. Solve the KdV equation for each ω up to a fixed time t to obtain $\mathcal{S}_t f_N$. The limit in N exists [KT1] in $H^{-1}(\mathbb{T})$, for almost every value of ω , and is again a white noise.

2. It follows immediately that

$$\hat{\mathcal{S}}_t : L^2(Q_0) \to L^2(Q_0), \quad (\hat{\mathcal{S}}_t \Phi)(f) = \Phi(\bar{\mathcal{S}}_t f) \tag{1.7}$$

are a group of *unitary* transformations of $L^2(Q_0)$, defining a continuous Markov process $u(t)$, $t \in (-\infty, \infty)$ on $H^{-1}(\mathbb{T})$ with Gaussian white noise one dimensional marginals, invariant under time+space inversions. The correlation functions $S(x,t) = \int f(0) \bar{\mathcal{S}}_t f(x) dQ_0$ may have an interesting structure.

3. A similar result holds without the mean zero condition, but now the mean $m = \int_{\mathbb{T}} u$ is distributed not as an independent Gaussian, but as one conditioned to have $m \geq -\lambda_0(u)$ where $\lambda_0(u)$ is the principal eigenvalue of $-\frac{d^2}{dx^2} + u$. Since the addition of constants produces a trivial rotation in the KdV equation it seems more natural to consider the mean zero case.

4. Q_0 is certainly *not* the only invariant measure for KdV. The Gibbs measure formally written as $Z^{-1} 1 \left(\int_{\mathbb{T}} u^2 \leq K \right) e^{-\mathcal{H}_2}$ where

$$\mathcal{H}_2(u) = -\int_{\mathbb{T}} u^3 - \frac{1}{2} u_x^2 \tag{1.8}$$

is known to be invariant [Bo]. Note that (after subtraction of the mean) this Gibbs measure is supported on a set of Q_0-measure 0. Q_0 is also a Gibbs measure, corresponding to the Hamiltonian

$$\mathcal{H}_1(u) = \int_{\mathbb{T}} u^2. \tag{1.9}$$

The existence of two Gibbs measures corresponds to the bihamiltonian structure of KdV: It can be written

$$\dot{u} = J_i \frac{\delta \mathcal{H}_i}{\delta u}, \qquad i = 1, 2 \tag{1.10}$$

with symplectic forms $J_1 = \partial_x^3 + 4u\partial_x + 2\partial_x u$ and $J_2 = \partial_x$. Because of all the conservation laws of KdV, there are many other invariant measures as well.

5. We were led to Theorem 1.2 after noticing that the discretization of KdV used by Kruskal and Zabusky in the numerical investigation of solitons,

$$\dot{u}_i = (u_{i+1} + u_i + u_{i-1})(u_{i+1} - u_{i-1}) - (u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}), \tag{1.11}$$

preserves discrete white noise (independent Gaussians mean 0 and variance $\sigma^2 > 0$). The invariance follows from two simple properties of the special discretization (1.11). First of all $\dot{u}_i = b_i$ preserves Lebesgue measure whenever $\nabla \cdot b = \sum_i \partial_i b_i = 0$, and (1.11) is of this form. Furthermore, it is easy to check (though something of a miracle) that $\sum_i u_i^2$ is invariant under (1.11). Hence $Z^{-1} e^{-\frac{1}{2\sigma^2} \sum_i u_i^2} \prod du_i$ is also invariant.

Note that the discretization (1.11) is *not* completely integrable, and we are not aware of a completely integrable discretization which does conserve discrete white noise. For example, consider the following family of completely integrable discretizations of KdV, depending on a real parameter α [AL];

$$\begin{aligned}\dot{u}_i &= (1-\alpha u_i)\{-\alpha u_{i-1}(u_{i-2}-u_i) - \alpha(u_{i-1}+2u_i+u_{i+1})(u_{i-1}-u_{i+1}) \\ &\quad -\alpha u_{i+1}(u_i - u_{i+2}) + u_{i-2} - 2u_{i-1} + 2u_{i+1} - u_{i+2}\}. \end{aligned} \tag{1.12}$$

They conserve Lebesgue measure by the Liouville theorem. We want $\alpha \neq 0$; otherwise the quadratic term of KdV is not represented. In that case the conserved quantity analagous to $\int_{\mathbb{T}} u^2$ is

$$\mathcal{Q} = \sum_i u_i^2 + 2u_i u_{i+1}. \tag{1.13}$$

But $\mathcal{Q}$ is non-definite, and hence the corresponding measure $e^{-\mathcal{Q}} \prod_i du_i$ cannot be normalized to make a probability measure.

6. At a completely formal level the proof proceeds as follows. Note first of all that the flow generated by $u_t = u_{xxx}$ is easily solved and seen to preserve white noise. So consider the Burgers' flow $u_t = 2uu_x$.

$$\partial_t \int f(u(t))e^{-\int u^2} = \int \langle \frac{\delta f}{\delta u}, u_t\rangle e^{-\int u^2} = \int \langle \frac{\delta f}{\delta u}, (u^2)_x\rangle e^{-\int u^2} = -\int f \langle \frac{\delta}{\delta u}(u^2)_x e^{-\int u^2}\rangle \tag{1.14}$$

and

$$\langle \frac{\delta}{\delta u}(u^2)_x e^{-\int u^2}\rangle = \langle (2u_x - (u^2)_x 2u)\rangle e^{-\int u^2}. \tag{1.15}$$

The last term vanishes because $(u^2)_x 2u = \frac{2}{3}(u^3)_x$ and because of periodic boundary conditions any exact derivative integrates to zero: $\langle f_x \rangle = \int_0^1 f = 0$.

Such an argument is known in physics [S]. Note that the problem is subtle, and requires an appropriate interpretation. In fact the result is not correct for the standard mathematical interpretation of the Burgers' flow as the limit as $\epsilon \downarrow 0$ of $u_t^\epsilon = 2u^\epsilon u_x^\epsilon + \epsilon u_{xx}^\epsilon$, as can be checked with the Lax-Oleinik formula. On the other hand, the argument is rigorous for (1.11).

2. Invariant measures for mKdV on $\mathbb{T}$

Let P_0 denote Wiener measure on $\phi \in C(\mathbb{T})$ conditioned to have $\int_{\mathbb{T}} \phi = 0$. It can be derived from the standard circular Brownian motion P on $C(\mathbb{T})$ defined as follows: Condition a standard Brownian motion $\beta(t)$, $t \in [0,1]$ starting at $\beta(0) = x$ to have $\beta(1) = x$ as well, and now distribute x on the real line according to Lebesgue measure. P_0 is obtained from P by conditioning on $\int_{\mathbb{T}} \phi = 0$.

Define $P_0^{(4)}$ to be the measure absolutely continuous to P_0 given by

$$P_0^{(4)}(B) = Z^{-1} \int_B J(\phi) e^{-\frac{1}{2}\int_{\mathbb{T}} \phi^4} dP_0, \tag{2.1}$$

for Borel sets $B \subset C(\mathbb{T})$ where Z is the normalizing factor to make $P_0^{(4)}$ a probability measure and

$$J(\phi) = (2\pi)^{-1/2} K(\phi) K(-\phi) e^{\frac{1}{2}\left(\int \phi^2\right)^2} \tag{2.2}$$

where

$$K(\phi) = \int_0^1 e^{2\Phi(x)} dx \tag{2.3}$$

and

$$\Phi(x) = \int_0^x \phi(y) dy. \tag{2.4}$$

For smooth g and $-\infty < t < \infty$, let $\phi(t) = \mathcal{M}_t g$ denote the (smooth) solution of the modified KdV (mKdV) equation,

$$\phi_t - 6\phi^2 \phi_x + \phi_{xxx} = 0, \qquad \phi(0) = g. \tag{2.5}$$

Theorem 2.1. *(Kappeler and Topalov* [KT3]*) $\mathcal{M}_t$ extends to a continuous group of nonlinear evolution operators*

$$\bar{\mathcal{M}}_t : L^2(\mathbb{T}) \to L^2(\mathbb{T}). \tag{2.6}$$

Let

$$H(\phi) = \frac{1}{2} \int_{\mathbb{T}} \phi^4 + \phi_x^2. \tag{2.7}$$

mKdV can be written in Hamiltonian form,

$$\phi_t = \partial_x \frac{\delta H}{\delta \phi}. \tag{2.8}$$

$P_0^{(4)}$ gives rigorous meaning to the weighted Gibbs measure $J(\phi)e^{-H(\phi)}$ on $\int \phi = 0$.

Theorem 2.2. *(Bourgain* [Bo]*) $P_0^{(4)}$ is invariant for mKdV,*

$$\bar{\mathcal{M}}_t^* P_0^{(4)} = P_0^{(4)}. \tag{2.9}$$

Proof. In fact what is proven in [Bo] is that $Z^{-1}e^{-\frac{1}{2}\int_{\mathbb{T}} \phi^4} dP$ is invariant for mKdV. The main obstacle at the time was a lack of well-posedness for mKdV on the support $H^{1/2-}$ of the measure. This statement follows with less work once one has the results of Kappeler and Topalov proving well-posedness on a larger set (Theorem 2.1).

We have in addition to show that $J(\phi)$ is a conserved quantity for mKdV. It is well known that $\int_{\mathbb{T}} \phi^2$ is preserved. So the problem is reduced to showing that $K(\phi)$ and $K(-\phi)$ are conserved. Let $\phi(t)$ be a smooth solution of mKdV. Note that

$$\partial_t \Phi = 2\phi^3 - \phi_{xx}. \tag{2.10}$$

Hence

$$\partial_t K = 2 \int_0^1 (2\phi^3 - \phi_{xx}) e^{2\Phi(x)} dx. \tag{2.11}$$

But integrating by parts we have, since ϕ is periodic and $\Phi_x = \phi$,

$$\int_0^1 \phi_{xx} e^{2\Phi(x)} dx = -\int_0^1 2\phi_x \phi e^{2\Phi(x)} dx = -\int_0^1 (\phi^2)_x e^{2\Phi(x)} dx = \int_0^1 2\phi^3 e^{2\Phi(x)} dx. \tag{2.12}$$

Therefore $\partial_t K(\phi(t)) = 0$. One can easily check with the analogous integration by parts that $\partial_t K(-\phi(t)) = 0$.

Now suppose $\bar{\mathcal{M}}_t \phi = \phi(t)$ with $\phi \in L^2(\mathbb{T})$. From Theorem 2.1 we have smooth ϕ_n with $\phi_n \to \phi$ and $\phi_n(t) \to \phi(t)$ in $L^2(\mathbb{T})$.

$$K(\phi(t)) - K(\phi) = [K(\phi(t)) - K(\phi_n(t))] - [K(\phi_n) - K(\phi)] \tag{2.13}$$

so if K is a continuous functions on $L^2(\mathbb{T})$ then $K(\phi)$ and $K(-\phi)$ are conserved by $\bar{\mathcal{M}}_t$. To prove that K is continuous simply note that

$$|K(\phi) - K(\psi)| = |\int_0^1 e^{2\int_0^x \phi} [e^{2\int_0^x \psi - \phi} - 1] dx| \le e^{2\|\phi\|_{L^2(\mathbb{T})}} [e^{2\|\psi-\phi\|_{L^2(\mathbb{T})}} - 1]. \tag{2.14}$$

∎

3. The Miura Transform on $L^2_0(\mathbb{T})$

The Miura transform $\phi \mapsto \phi_x + \phi^2$ maps smooth solutions of mKdV to smooth solutions of KdV. It is basically two to one, and not onto. But this is mostly a matter of the mean $\int_{\mathbb{T}} \phi$. Since the mean is conserved in both mKdV and KdV, it is more natural to consider the map corrected by subtracting the mean. The corrected Miura transform is defined for smooth ϕ by,

$$\mu(\phi) = \phi_x + \phi^2 - \int_{\mathbb{T}} \phi^2. \tag{3.1}$$

Let $L^2_0(\mathbb{T})$ and $H^{-1}_0(\mathbb{T})$ denote the subspaces of $L^2(\mathbb{T})$ and $H^{-1}(\mathbb{T})$ with $\int_{\mathbb{T}} \phi = 0$.

Theorem 3.1. *(Kappeler and Topalov* [KT2]*) The corrected Miura transform μ extends to a continuous map*

$$\bar{\mu} : L^2_0 \to H^{-1}_0 \tag{3.2}$$

which is one to one and onto. $\bar{\mu}$ takes solutions ϕ of mKdV (2.5) on $L^2(\mathbb{T})$, to solutions $u = \mu(\phi)$ of KdV (1.1) on $H^{-1}(\mathbb{T})$;

$$\bar{\mathcal{S}}_t \bar{\mu} = \bar{\mu} \bar{\mathcal{M}}_t. \tag{3.3}$$

Remark. The Ricatti map is given by

$$r(\phi, \lambda) = \phi_x + \phi^2 + \lambda. \tag{3.4}$$

Note that Kappeler and Topalov use the term Ricatti map for $\mu = r(\phi, -\int_{\mathbb{T}} \phi^2)$.

4. The Miura Transform on Wiener Space

Theorem 4.1. *(Cambronero and McKean* [CM]*) The corrected Miura transform $\bar{\mu}$ maps $P^{(4)}_0$ into mean zero white noise Q_0;*

$$\mu^* P^{(4)}_0 = Q_0. \tag{4.1}$$

Proof. Let $\hat{P}^{(4)}_0$ be given by

$$\hat{P}^{(4)}(B) = \frac{1}{\sqrt{2\pi}} \int_{(\phi,\lambda)\in B} K(\phi)K(-\phi) e^{-\frac{1}{2}\int_{\mathbb{T}}(\phi^2+\lambda)^2} dP d\lambda \tag{4.2}$$

where B is a Borel subset of $C(\mathbb{T}) \times \mathbb{R}$. Let $\hat{r} = (r, \int_{\mathbb{T}} \phi)$. Let $\hat{Q}$ on $C(\mathbb{T}) \times \mathbb{R}$ be given by $\hat{Q} = Q \times$ Lebesgue measure. What is actually proved in [CM] is that

$$\hat{r}^* \hat{P}^{(4)} = \hat{Q}. \tag{4.3}$$

(4.1) is obtained by conditioning on $\lambda = -\int_{\mathbb{T}} \phi^2$ and $\int_{\mathbb{T}} \phi = 0$. ∎

Remark. There is a simple heuristic argument explaining (4.1). Formally

$$dP^{(4)}_0 = Z_1^{-1} K(\phi)K(-\phi) e^{-\frac{1}{2}\int_0^1 (\phi^2 + \phi' - \int_0^1 \phi^2)^2} dF(\phi), \qquad dQ_0 = Z_2^{-1} e^{-\frac{1}{2}\int_0^1 u^2} dF(u) \tag{4.4}$$

where F is the (mythical) flat measure on $\int_0^1 \phi = 0$. Note that in the exponent of $dP^{(4)}_0$ we have assumed that integration by parts gives $\int_0^1 \phi^2 \phi' = 0$. Since the corrected Miura transform $u = \phi^2 + \phi' - \int_0^1 \phi^2$ the only mystery is the form of the Jacobian $CK(\phi)K(-\phi)$. Let D be the map $Df = f'$ and ϕ stand for the map of multiplication by ϕ with a subtraction to make the result mean zero, $\phi f = \phi \cdot f - \int_0^1 \phi \cdot f$. The Jacobian is then,

$$f(\phi) = \det(1 + 2\phi D^{-1}) = \exp\{\operatorname{Tr} \log(1 + 2\phi D^{-1})\} \tag{4.5}$$

For fixed $x, y \in \mathbb{T}$ let $\partial_{xy} = \frac{\partial}{\partial(\phi(y)-\phi(x))}$, i.e. the Gâteaux derivative in the direction $\delta_y - \delta_x$, $\partial_{xy} F(\phi) = \lim_{\epsilon\to 0} \epsilon^{-1}(F(\phi + \epsilon(\delta_y - \delta_x)) - F(\phi)$. We have

$$\partial_{xy} \log f(\phi) = \partial_{xy} \mathrm{Tr} \log(1 + 2\varphi D^{-1}) = \mathrm{Tr}[\{\partial_{xy}(1 + 2\phi D^{-1})\}\{1 + 2\phi D^{-1})^{-1}\}]. \tag{4.6}$$

If we let $G(x, y)$ denote the Green function of $D + 2\phi$ this gives $\partial_{xy} \log f(\phi) = 2[G(y, y) - G(x, x)]$.. It is not hard to compute the Green function with the result that

$$\partial_{xy} \log f(\phi) = \frac{2\int_x^y e^{-2\Phi}}{\int_0^1 e^{-2\Phi}} - \frac{2\int_x^y e^{-2\Phi}}{\int_0^1 e^{-2\Phi}}. \tag{4.7}$$

The argument is completed by a straightforward verification that this is satisfied by $f(\phi) = K(\phi)K(-\phi)$.

The heuristic argument can be made rigorous by taking finite dimensional approximations where this set of equations actually identifies the determinant. Since the computations become exactly those of [CM], we do not repeat them here.

5. Proof of Theorem 1.2

$$\bar{\mathcal{S}}_t^* Q_0 \overset{\text{Thm 4.1}}{=} \bar{\mathcal{S}}_t^* \mu^* P_0^{(4)} \overset{\text{Thm 3.1}}{=} \mu^* \bar{\mathcal{M}}_t^* P_0^{(4)} \overset{\text{Thm 2.2}}{=} \mu^* P_0^{(4)} \overset{\text{Thm 4.1}}{=} Q_0 \tag{5.1}$$

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