LECTURE 12

Review of Radon-Nikodym. Let P and $\widetilde{P}$ be two probability measures on $(\Omega, \mathcal{F})$.

- $\widetilde{P} << P$ if $\widetilde{P}(A) = 0$ whenever $P(A) = 0$. We say $\widetilde{P}$ is *absolutely continuous* with respect to P.
- If there exist disjoint sets A and B such that $P(A) = 1$ and $\widetilde{P}(B) = 1$ then we say P and $\widetilde{P}$ are *mutually singular* and write $P \perp \widetilde{P}$.
- If $\widetilde{P} << P$ and $P << \widetilde{P}$ then we say P and $\widetilde{P}$ are *equivalent*.

Radon-Nikodym Theorem. *If $\widetilde{P} << P$ then there exists a non-negative random variable Z such that for any bounded random variable X,*

$$\widetilde{E}[X] = E[XZ]$$

where $\widetilde{E}$ is expectation with respect to $\widetilde{P}$ and E is expectation with respect to P.

Z is often denoted $\frac{d\widetilde{P}}{dP}$.

Example. Suppose that P is a probability measure on the real line $\mathbf{R}$ such that $P(A) = 0$ whenever the Lebesgue measure of A is zero. Then by the Radon-Nikodym theorem there exists a probability density function $f(x)$ such that

$$P(A) = \int_A f(x)dx.$$

(Note that this does not follow directly from the way we stated the theorem. What has to be done?)

Cameron-Martin-Girsanov formula. *Let $B(t)$, $0 \le t \le T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}(t)$, $0 \le t \le T$. Let $\theta(t)$, $0 \le t \le T$ be a process adapted to $\mathcal{F}(t)$, $0 \le t \le T$ and satisfying $E[e^{\frac{1}{2}\int_0^T \theta^2(t)dt}] < \infty$. Define*

$$\widetilde{B}(t) = \int_0^t \theta(u)du + B(t)$$

and

$$Z(t) = e^{-\int_0^t \theta(u)dB(u) - \frac{1}{2}\int_0^t \theta^2(u)du}.$$

Define a new probability measure $\widetilde{P}$ by $\widetilde{P}(A) = \int_A Z(T)dP$ forall $A \in \mathcal{F}$, or equivalently, for every bounded random variable X,

$$\widetilde{E}[X] = E[XZ(T)].$$

Then under $\widetilde{P}$, $\widetilde{B}(t)$, $0 \le t \le T$ is a Brownian motion.

Bayes rule. If X is $\mathcal{F}(t)$-measurable then for $0 \le s < t$,

$$\widetilde{E}[X \mid \mathcal{F}(s)] = \frac{1}{Z(s)} E[XZ(t) \mid \mathcal{F}(s)].$$

Proof. Recall that there are two requirements from the definition of conditional expectation:

1. $\widetilde{E}[X \mid \mathcal{F}(s)]$ should be measurable with respect to $\mathcal{F}(s)$.
2. We should have $\int_A \widetilde{E}[X \mid \mathcal{F}(s)]d\widetilde{P} = \int_A X d\widetilde{P}$ for any $A \in \mathcal{F}(s)$.

So we just have to check that $\frac{1}{Z(s)}E[XZ(t) \mid \mathcal{F}(s)]$ works in 1. and 2. It is clearly $\mathcal{F}(s)$-measurable, by definition of conditional expectation so 1. is easy. To check 2. let $A \in \mathcal{F}(s)$. We can write

$$\int_A \frac{1}{Z(s)}E[XZ(t) \mid \mathcal{F}(s)]d\widetilde{P} = \widetilde{E}[\mathbf{1}_A \frac{1}{Z(s)}E[XZ(t) \mid \mathcal{F}(s)]].$$

Now by definition, $\widetilde{E}[X] = E[XZ(s)]$ for any $\mathcal{F}(s)$-measurable X, so

$$\widetilde{E}[\mathbf{1}_A \frac{1}{Z(s)}E[XZ(t) \mid \mathcal{F}(s)]] = E[\mathbf{1}_A E[XZ(t) \mid \mathcal{F}(s)]].$$

But since $\mathbf{1}_A$ is measurable with respect to $\mathcal{F}(s)$, it can be taken into the inside expectation. But then we just have a double expectation which can be rewritten as $E[\mathbf{1}_A XZ(t)]$ which by definition is just $\widetilde{E}[\mathbf{1}_A X] = \int_A X d\widetilde{P}$ as desired.

Risk-neutral valuation. We have a stock and a risk-free bond. The stock price $S(t)$ is assumed to satisfy

$$dS(t) = S(t)\left\{\mu(t)dt + \sigma(t)dB(t)\right\} \tag{1}$$

where $B(t)$ is a Brownian motion on $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}(t)$. $\mu(t)$ and $\sigma(t)$ and the interest rate $r(t)$ are all adapted to $\mathcal{F}(t)$. The investor starts with $X(0)$ and at time t has $\Delta(t)S(t)$ in stock and $X(t) - \Delta(t)S(t)$ in bonds. $\Delta(t)$ is also assumed to be adapted to $\mathcal{F}(t)$. The wealth process $X(t)$ satisfies

$$dX(t) = \Delta(t)dS(t) + r(t)(X(t) - \Delta(t)S(t))dt. \tag{2}$$

The value of a dollar at time t in present dollars is $\frac{1}{\beta(t)} = e^{-\int_0^t r(u)du}$ so all assets should really be discounted by this factor.

A risk-neutral measure is a probability measure $\widetilde{P}$ equivalent to P under which all tradable assets are martingales after discounting.

If there exists a risk-neutral measure then the value at time s of a hedgable contingent claim paying $V(T)$ at time T must be

$$\beta(s)\widetilde{E}[\beta(T)^{-1}V(T) \mid \mathcal{F}(s)]$$

for if $\Delta(t)$ is the hedge then we have $X(T) = V(T)$ and $\beta^{-1}(t)X(t)$ is a martingale under $\widetilde{P}$ so $\beta^{-1}(s)X(s) = \widetilde{E}[\beta^{-1}(T)V(T) \mid \mathcal{F}(s)]$.

To find the risk-neutral measure we rewrite the equation for $S(t)$ as

$$d\left\{\frac{S(t)}{\beta(t)}\right\} = \frac{\sigma(t)S(t)}{\beta(t)}\left\{\theta(t)dt + dB(t)\right\}$$

where

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)} = \text{ the market price of risk.} \tag{3}$$

Let

$$\widetilde{B}(t) = \int_0^t \theta(u)du + B(t) \tag{4}$$

so that the equation becomes

$$d\left\{\frac{S(t)}{\beta(t)}\right\} = \frac{\sigma(t)S(t)}{\beta(t)}d\widetilde{B}(t).$$

Define $\widetilde{P}(A) = \int_A Z(T)dP$ where $Z(T) = e^{-\int_0^T \theta(u)dB(u) - \frac{1}{2}\int_0^T \theta^2(u)du}$ as before. Then by Cameron-Martin-Girsanov, $\widetilde{B}(t)$ is a Brownian motion under $\widetilde{P}$ and hence, since it is a stochastic integral, $\beta^{-1}(t)S(t)$ is a martingale with respect to $\widetilde{P}$. Note that $d\{\beta^{-1}X(t)\} = \Delta(t)d\{\beta^{-1}S(t)\}$ so it is automatically a martingale as well, so all tradable assets are martingales after discounting, as desired.

Example (European call, time and price dependent volatility). Let $V(T) = (S(T) - K)_+$, and $\sigma(t,\omega) = \sigma(t, S(t))$, $\mu(t,\omega) = \mu(t, S(t))$. The value at time 0 is

$$V(0) = \widetilde{E}[e^{-\int_0^T r(u)du}(S(T) - K)_+]$$

where $\widetilde{E}$ is the risk-neutral measure. In other words, $S(t)$ evolves according to the stochastic differential equation $d\{\beta^{-1}(t)S(t)\} = \beta^{-1}(t)\sigma(t)S(t)d\widetilde{B}(t)$ where $\widetilde{B}(t)$ is a Brownian motion under $\widetilde{P}$. This is of course equivalent to

$$dS(t) = S(t)\{r(t)dt + \sigma(t)d\widetilde{B}(t)\}.$$

By the Feynman-Kac formula we know that

$$u(T,x) = \widetilde{E}_x[e^{-\int_0^T r(u)du}(S(T) - K)_+]$$

is the solution of the equation

$$\frac{\partial u}{\partial T} = \frac{1}{2}x^2\sigma^2(T,x)\frac{\partial^2 u}{\partial x^2} + xr(T)\frac{\partial u}{\partial x} - r(T)u, \qquad u(0,x) = (x - K)_+$$

and our answer $V(0) = u(T, S(0))$.

Only in special cases (eg. Black-Scholes) is the equation explicitly solvable. Otherwise you may have to use numerical methods.

Martingale representation theorem. *Let $B(t)$ be a Brownian motion with filtration $\mathcal{F}(t)$ and suppose that $X(t)$ is a martingale with respect to $\mathcal{F}(t)$. Then there exists an adapted process $\delta(t)$ such that*

$$X(t) = X(0) + \int_0^t \delta(u)dB(u).$$

In particular it follows that the process $X(t)$ has continuous sample paths.

We now know that stochastic integrals with respect to Brownian motion are martingales with respect to the Brownian filtration, and vice-versa, martingales with respect to the Brownian filtration arise as stochastic integrals.

Application to risk-neutral valuation. Suppose we have a stock price satisfying equation 1. We define the risk-neutral measure as in equations 3 and 4. Let $V(T)$ be measurable with respect to $\mathcal{F}(T)$. We want to use the martingale representation theorem to construct a hedge. For $0 \leq t \leq T$ let

$$M(t) = \widetilde{E}[\beta^{-1}(T)V(T) \mid \mathcal{F}(t)].$$

Then $M(t)$ is a martingale under $\widetilde{P}$ with respect to the filtation $\mathcal{F}(t)$. By the martingale representation theorem there exists an adapted process $\gamma(t)$ so that

$$M(t) = M(0) + \int_0^t \gamma(u) d\widetilde{B}(u).$$

Choose $\Delta(u)$ to satisfy $\Delta(u)\beta^{-1}(u)\sigma(u)S(u) = \gamma(u)$ and let $X(t) = \beta(t)M(t)$. Then $X(t)$ satisfies an equation of the form 2 and $X(T) = V(T)$, so it is a hedge.

Problems.

1. i. If P and $\widetilde{P}$ are equivalent and $\frac{d\widetilde{P}}{dP} = Z$ show that $\frac{dP}{d\widetilde{P}} = \frac{1}{Z}$.

Let $P_x^{a,b}$ denote the probability measure on $C([0,T])$ corresponding to the solution of the stochastic differential equation

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, \qquad X(0) = x$$

where $a = \sigma\sigma^T$.

ii. Let $b_1 \neq b_2$. Write expressions for $\frac{dP_x^{a,b_1}}{dP_x^{a,b_2}}$ and $\frac{dP_x^{a,b_2}}{dP_x^{a,b_1}}$ using the Cameron-Martin-Girsanov formula.

iii. Resolve the apparent contradiction between the answer to *ii* and *i*.

2. Let a stock evolve according to $dS = \mu S dt + \sigma S dB$ and a bond evolve as $d\beta = r\beta dt$ where σ, μ and r are adapted to the Brownian filtration. Let $V(T)$ be a random variable measurable with respect to $\mathcal{F}(T)$ representing the payoff at time T of a contingent claim. As we know the value at time t is given by

$$V(t) = \beta(t)\widetilde{E}[\beta^{-1}(T)V(T) \mid \mathcal{F}(t)]$$

where $\widetilde{P}$ is the risk-neutral measure. Show that this can be rewritten in the form

$$\zeta^{-1}(t)E[\zeta(T)V(T) \mid \mathcal{F}(t)]$$

and find an explicit expression for the process $\zeta(t)$.

3. Let $\alpha(x) = (\alpha_1(x), \dots, \alpha_n(x))$ be a smooth function from $\mathbf{R}^n$ to $\mathbf{R}^n$. Consider the partial differential equation for $x \in \mathbf{R}^n$, and $t > 0$,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} \alpha_i(x)\frac{\partial u}{\partial x_i}, \qquad u(0,x) = f(x).$$

   i. Use the Girsanov theorem to show that the solution is

$$u(t,x) = E_x\left[e^{\int_0^t \alpha(B(s))dB(s) - \frac{1}{2}\int_0^t |\alpha(B(s))|^2 ds} f(B(t))\right].$$

   ii. Suppose that $\alpha(x) = \nabla\gamma(x)$ for some function $\gamma : \mathbf{R}^n \to \mathbf{R}$. Use Ito's formula to show that in this case

$$u(t,x) = e^{-\gamma(x)} E_x\left[e^{\gamma(B(t)) - \frac{1}{2}\int_0^t \left(\nabla\gamma^2(B(s)) + \Delta\gamma(B(s))\right) ds} f(B(t))\right].$$

   iii. Use the Feynman-Kac formula to show that $v(t,x) = e^{\gamma(x)}u(t,x)$ is the solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \frac{1}{2}(\nabla\gamma^2 + \Delta\gamma)v.$$