LECTURE 11

Two notions of uniqueness for stochastic differential equations. 1. *Pathwise uniqueness* of the stochastic differential equation $dX = bdt + \sigma dB$, $X(0) = x$ means that if $X_1(t)$ and $X_2(t)$ are two solutions with the same Brownian motion, then $X_1(t) = X_2(t)$ for all $t > 0$ almost surely. 2. *Uniqueness in the sense of law* means that if $X_1(t)$ is a solution of $dX_1(t) = b(t, X_1(t))dt + \sigma(t, X_1(t))dB_1(t)$, $X_1(0) = x$ and $X_2(t)$ is a solution of $dX_2(t) = b(t, X_2(t))dt + \sigma(t, X_2(t))dB_2(t)$, $X_2(0) = x$, then X_1 and X_2 have the same law - that is the collection of probabilities $P(X(\cdot) \in \mathcal{A})$ where $\mathcal{A}$ is a (measurable) subset of the space of continuous functions.

Pathwise uniqueness implies uniqueness in law but not the other way around.

Markov property of solutions of stochastic differential equations. Let $X(t)$ be the unique (in law) solution after time s of the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), \quad t > s \qquad X(s) = x.$$

One denotes the expectation with respect to this process as $E_{s,x}[\cdot]$. Sometimes it is useful to note explicitly the dependence on the initial conditions and people write things like $X_{s,x}(t)$.

The solutions are Markov processes, i.e. for any nice function $f(x)$, and any $t \geq s$,

$$E_{0,x}[f(X(t)) \mid \mathcal{F}(s)] = E_{s,X(s)}[f(X(t))].$$

Proof. This follows from the uniqueness which guarantees that for $t > s$, $X_{s,X(s)}(t) = X_{0,x}(t)$ because otherwise they would represent two distinct solutions to the sde starting at $X(s)$ at time s.

Transition probabilities. Since $X(t)$ is a Markov process, its law is determined by the transition probabilities,

$$p(s, x, t, A) = P(X(t) \in A \mid X(s) = x), \qquad t > s.$$

Another way to write this is

$$E_{s,x}[f(X(t))] = \int f(y)p(s, x, t, dy)$$

for any function f.

Recall that transition probabilities always satisfy the Chapman-Kolmogorov equations $p(s, x, t, A) = \int p(s, x, u, dy)p(u, y, t, A)$ for any $s < u < t$, and that all finite dimensional distributions of the process can be written in terms of them,

$$P(X(t_1) \in A_1, \ldots, X(t_n) \in A_n) = \int_{A_n} \cdots \int_{A_1} p(0, x, t_1, dx_1) \cdots p(t_{n-1}, x_{n-1}, t_n, dx_n).$$

Except in degenerate cases, the transition probabilities of a diffusion process will have densities

$$p(s, x, t, A) = \int_A p(s, x, t, y)dy.$$

Note the abuse of notation.

Backward and forward equations. Suppose that $X(t)$ is the unique (in law) solution of a stochastic differential equation $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$, $X(0) = x$. We know that the solution is a Markov process. How do we find the transition probabilities? Recall our old friend the operator

$$L_t f = \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(t,x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x) \frac{\partial f}{\partial x_i}$$

where $a = \sigma\sigma^T$. Its adjoint is

$$L_t^* f(x) = \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} (a_{i,j}(t,x) f(x)) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} (b_i(t,x) f(x)).$$

Proposition. The transition probabilities satisfy the *forward* or *Fokker-Planck* equations

$$\frac{\partial}{\partial t} p(s,x,t,y) = L_t^* p(s,x,t,y)$$

where the operator L_t^* is acting on the y variable. x and s are just fixed parameters playing no part in the partial differential equation. They only appear in the initial condition

$$\lim_{t \downarrow s} p(s,x,t,y) = \delta(y-x).$$

Proof. Let $f(t,x)$ be any smooth function. By Ito's formula

$$E_{s,x}[f(T, X(T))] - f(s,x) = E_{s,x} \left[\int_s^T \left\{ \frac{\partial f}{\partial t} + L_t f \right\} (t, X(t)) dt \right].$$

In terms of the transition probabilities we can write this as

$$\int f(T,y) p(s,x,T,y) dy - f(s,x) = \int_s^T \int \left\{ \frac{\partial f}{\partial t} + L_t f \right\} (t,y) p(s,x,t,y) dy dt.$$

Integrating by parts, once in the t variable and twice in the y variable we get

$$\int_s^T \int f(t,y) \{ -\frac{\partial}{\partial t} + L_t^* \} p(s,x,t,y) dy dt = 0.$$

If $\int_s^T \int f(t,y) g(t,y) dy dt = 0$ for every smooth function, then $g(t,y) = 0$ for $s \le t \le T$. This gives the forward equation.

Proposition. The transition probabilities satisfy the *backward* or *Kolmogorov* equations

$$-\frac{\partial}{\partial s} p(s,x,t,y) = L_s p(s,x,t,y).$$

Here the operator L_s acts on the variable x and the t and y are parameters appearing only in the final condition

$$\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x).$$

Proof. Let $f(x)$ be any smooth function $u(s,x)$ be the unique solution of $-\frac{\partial}{\partial s}u = L_s u$ for $0 \le s < t$ with $u(t,x) = f(x)$. By Ito's formula, $u(s,X(s))$ is a martingale up to time t. Taking expectation

$$E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))].$$

By the final condition, the right hand side is $\int f(z)p(s,x,t,z)dz$. The left hand side is just $u(s,x)$. So

$$u(s,x) = \int f(z)p(s,x,t,z)dz.$$

Letting $f_n(z)$ be a sequence of smooth functions tending to $\delta(y-z)$ we get in the limit that p satisfy the backward equations.

Examples.

1. **Brownian motion in 1 dimension.** The operator $L = \frac{1}{2}\frac{\partial^2}{\partial x^2}$. The forward equation is $\frac{\partial}{\partial t}p(s,x,t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}p(s,x,t,y)$ with initial condition $\lim_{t\downarrow s} p(s,x,t,y) = \delta(y-x)$ and the backward equation is $-\frac{\partial}{\partial s}p(s,x,t,y) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(s,x,t,y)$ with final condition $\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x)$. Both are of course satisfied by the heat kernel $p(s,x,t,y) = \frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-y)^2}{2(t-s)}}$.
2. **Brownian motion in d dimensions.** The operator $L = \frac{1}{2}\Delta = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$. The forward equation is $\frac{\partial}{\partial t}p(s,x,t,y) = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial y_i^2}p(s,x,t,y)$ with initial condition $\lim_{t\downarrow s} p(s,x,t,y) = \delta(y-x)$ and the backward equation is $-\frac{\partial}{\partial s}p(s,x,t,y) = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}p(s,x,t,y)$ with final condition $\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x)$. Both are satisfied by the heat kernel $p(s,x,t,y) = \frac{1}{(2\pi(t-s))^{d/2}}e^{-\frac{|x-y|^2}{2(t-s)}}$.
3. **Brownian motion with constant drift b and constant volatility σ^2.** Consider the constant coefficient stochastic differential equation

$$dX(t) = bdt + \sigma dB(t), \qquad X(0) = x.$$

   The solution is $X(t) = bt + \sigma B(t)$ so it is easy to check that

$$p(s,x,t,y) = \frac{1}{\sqrt{2\pi\sigma^2(t-s)}}e^{-\frac{(y-x-b(t-s))^2}{2\sigma^2(t-s)}}.$$

   You can also check that it satisfies the forward equation $\frac{\partial}{\partial t}p(s,x,t,y) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial y^2}p(s,x,t,y) - b\frac{\partial}{\partial y}p(s,x,t,y)$ and the backward equation $-\frac{\partial}{\partial s}p(s,x,t,y) = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(s,x,t,y) + b\frac{\partial}{\partial x}p(s,x,t,y)$.
4. **Ornstein-Uhlenbeck process.** For the stochastic differential equation $dX(t) = -\rho X(t)dt + dB(t)$ the corresponding Markov generator is $L = \frac{1}{2}\frac{\partial^2}{\partial x^2} - \rho x\frac{\partial}{\partial x}$. The

forward equation is $\frac{\partial}{\partial t}p(s,x,t,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}p(s,x,t,y) + \rho\frac{\partial}{\partial y}(yp(s,x,t,y))$. The backward equation is $-\frac{\partial}{\partial s}p(s,x,t,y) = \frac{1}{2}\frac{\partial^2}{\partial x^2}p(s,x,t,y) + \rho x\frac{\partial}{\partial x}p(s,x,t,y)$. Since the process is time-homogeneous (i.e. b and σ do not depend on time) the solution must be of the form $p(s,x,t,y) = p(t-s,x,y)$. Letting $v(t,x) = p(t, xe^{\rho t}, y)$ one gets the equation $e^{2\rho t}\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}$ which can be transformed into the standard heat equation by $\tau = (2\rho)^{-1}(1-e^{-2\rho t})$. Solving and undoing all these transformations one gets *Mehler's formula*,

$$p(t,x,y) = \frac{e^{-\frac{\rho(y-xe^{-\rho t})^2}{(1-e^{-2\rho t})}}}{\sqrt{2\pi\left(\frac{1-e^{-2\rho t}}{2\rho}\right)}}.$$

5. **Geometric Brownian motion.** The stochastic differential equation is $dX(t) = rX(t)dt + \sigma X(t)dB(t)$. The Markov generator is $L = \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x}$. Since the process is time-homogeneous the transition probabilities have the form $p(s,x,t,y) = p(t-s,x,y)$ and the backward equation becomes

$$\frac{\partial}{\partial t}p(t,x,y) = \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}p(t,x,y) + rx\frac{\partial}{\partial x}p(t,x,y), \qquad p(0,x,y) = \delta(y-x).$$

If you like, you can try to solve it. But in this case we actually know the solution of the stochastic differential equation is

$$X(t) = xe^{\sigma B(t) + (r-\frac{1}{2}\sigma^2)t}.$$

So

$$P_x(X(t)\in A) = P(xe^{\sigma B(t)+(r-\frac{1}{2}\sigma^2)t}\in A).$$

In other words, for fixed x and t, $p(t,x,y)$ is the density function of $\phi(B(t))$ where $\phi(y) = xe^{\sigma y + (r-\frac{1}{2}\sigma^2)t}$. Recall that if X is a random variable with density f the density of $\phi(X)$ is given by $(f\circ\phi^{-1})/\phi'$. Since $\phi^{-1}(y) = \sigma^{-1}\left(\log\frac{y}{x} - (r-\frac{1}{2}\sigma^2)t\right)$ we have

$$p(t,x,y) = \frac{1}{\sigma y\sqrt{2\pi t}}e^{-\frac{\left(\log\frac{y}{x} - (r-\frac{1}{2}\sigma^2)t\right)^2}{2t\sigma^2}}.$$

You can check it satisfies the backward equation if you like long calculations.

6. **Black-Scholes.** The price at time t of a European call option with maturity T and strike price K is

$$E[e^{-r(T-t)}(X(T)-K)_+ \mid \mathcal{F}(t)] = e^{-r(T-t)}E_{t,X(t)}[(X(T)-K)_+]$$

where $X(t)$ is a geometric Brownian motion $dX(t) = rX(t)dt + \sigma X(t)dB(t)$ where r is the riskless interest rate and σ is the stock volatility. Now

$$E_{t,x}[((X(T)-K)_+] = \int (y-K)_+ p(T-t,x,y)dy$$

where $p(t, x, y)$ are the transition probabilities of the geometric Brownian motion given above. It is not hard to write down the answer in terms of the error function

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy.$$

We get

$$\begin{aligned} E[e^{-r(T-t)}(X(T)-K)_+ \mid \mathcal{F}(t)] =& X(t)\Phi\left(\frac{\log\frac{X(t)}{K} + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- e^{-r(T-t)}K\Phi\left(\frac{\log\frac{X(t)}{K} + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Diffusion process as probability measure on $C([0,\infty))$. You can either think of Brownian motion as a collection of random variables $B(t)$, $t \geq 0$ on some probability space $(\Omega, \mathcal{F}, P)$, or as a probability measure P on the space $C([0,\infty))$ of continuous functions. If $\mathcal{A}$ is a (measurable) subset of continuous functions, then $P(\mathcal{A})$ is just the probability that a Brownian path falls in that subset.

The same holds for any diffusion. If $X(t)$ is the solution of the stochastic differential equation $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$, $X(0) = x$ then we can let $P_x^{a,b}$ denote the probability measure on the space of continuous functions with

$$P_x^{a,b}(\mathcal{A}) = Prob(X(\cdot) \in \mathcal{A}).$$

Here $a = \sigma\sigma^T$ and we label it by a and not σ because it really only depends on a. Also we use the notation $X(\cdot)$ to indicate that we are thinking of the whole function as opposed to $X(t)$ which is just the value of that function at t.

Relation of $P_x^{a,b}$ for different values of x. The measure $P_x^{a,b}$ is of course concentrated on paths starting at x, so if $x_1 \neq x_2$ then $P_{x_1}^{a,b} \perp P_{x_2}^{a,b}$ meaning they are concentrated on completely different subsets of $C([0,\infty))$.

Relation of $P_x^{a,b}$ for different values of a. Just as with Brownian motion, it is possible to show that the quadratic variation

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor 2^n T \rfloor} |X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})|^2 = \int_0^T a(t, X(t))dt$$

almost surely. Hence if $a_1 \neq a_2$, $P_x^{a_1,b} \perp P_x^{a_2,b}$.

Sometimes it may be useful to know how different they are. Suppose we solve the two constant coefficient sde's $dX_1(t) = bdt + \sigma_1 dB(t)$ and $dX_2(t) = bdt + \sigma_2 dB(t)$. We ask for the relative probability that $X_i(t_1) = x_1, \ldots X_i(t_n) = x_n$. From the formula we worked out we have

$$\begin{aligned} &\frac{P(X_1(t_1) \in dx_1, \ldots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \ldots, X_2(t_n) \in dx_n)} \\ &= \left(\frac{\sigma_2^2}{\sigma_1^2}\right)^{n/2} \exp\left\{-\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i - b(t_{i+1} - t_i))^2}{2(t_{i+1} - t_i)}\right\} \end{aligned}$$

which tells you that the relative probability at n times is roughly of size e^{cn}.

Relation of $P_x^{a,b}$ for different values of b (Cameron-Martin-Girsanov formula). Let's try the same thing with $dX_1(t) = b_1 dt + \sigma dB(t)$ and $dX_2(t) = b_2 dt + \sigma dB(t)$. Then $\frac{P(X_1(t_1)\in dx_1,\ldots,X_1(t_n)\in dx_n)}{P(X_2(t_1)\in dx_1,\ldots,X_2(t_n)\in dx_n)}$ is given by

$$\exp\left\{-\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i-b_1(t_{i+1}-t_i))^2-(x_{i+1}-x_i-b_2(t_{i+1}-t_i))^2}{2\sigma^2(t_{i+1}-t_i)}\right\}.$$

Rewriting the term in the exponential we get

$$-\sum_{i=0}^{n-1}\sigma^{-1}(b_2-b_1)\sigma^{-1}(x_{i+1}-x_i-b_2(t_{i+1}-t_i))-\frac{1}{2}\sum_{i=0}^{n-1}(\sigma^{-1}(b_2-b_1))^2(t_{i+1}-t_i).$$

Under P_x^{a,b_2}, $\sigma^{-1}(X(t)-b_2t)$ is a Brownian motion, so if we take limits, then under this measure we get

$$\frac{dP_x^{a,b_1}}{dP_x^{a,b_2}} = \exp\left\{-\int_0^t\sigma^{-1}(b_2-b_1)dB(s)-\frac{1}{2}\int_0^t(\sigma^{-1}(b_2-b_1))^2ds\right\}.$$

This is the *Cameron-Martin-Girsanov* formula. In particular the formula tells us that as long as this thing is integrable with respect to P_x^{a,b_2}, then $P_x^{a,b_1} << P_x^{a,b_2}$, i.e. if $P_x^{a,b_2}(\mathcal{A}) = 0$ then $P_x^{a,b_1}(\mathcal{A}) = 0$ as well.

Problems.

1. i. Show that $X(t) = (1-t)\int_0^t \frac{1}{1-s}dB(s)$ is the solution of

$$dX(t) = -\frac{X(t)}{1-t}dt + dB(t), \qquad 0\le t<1, \qquad X(0)=0.$$

   ii. Show that $\lim_{t\uparrow 1} X(t) = 0$ almost surely. $X(t)$ is called the *Brownian bridge*.
   iii. Show that $X(t)$ is Gaussian and calculate its mean and covariance.
   iii. For fixed constants a and b solve the stochastic differential equation

$$dX(t) = \frac{b-X(t)}{1-t}dt + dB(t), \qquad 0\le t<1 \qquad X(0)=a.$$

2. i. Solve the stochastic differential equation $dX(t) = \lambda X(t)dB(t)$, $X(0) = 1$.
   ii. For $n = 0, 1, \ldots$ define

$$H_n(t,x) = (-1)^n\frac{t^n}{n!}e^{\frac{x^2}{2t}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2t}}.$$

   These are called the *Hermite polynomials*.
   iii. Show that

$$e^{\lambda x-\frac{1}{2}\lambda^2 t} = \sum_{n=0}^{\infty}\lambda^n H_n(t,x).$$

iv. Show that the Hermite polynomials play the role for stochastic calculus that $x^n/n!$ play for ordinary calculus:

$$dH_{n+1}(t, B(t)) = H_n(t, B(t))dt.$$

3. Let $B(t)$ be Brownian motion. Use Ito's formula to find an expression for $E[B^n(t)]$ in terms of $E[B^{n-2}(t)]$. Use induction to compute $E[B^n(t)]$ for every positive integer n.
4. Let

$$C_K(s, \omega, t) = E[(X(t) - K)_+ \mid \mathcal{F}(s)]$$

where $X(t)$ is the solution of a stochastic differential equation $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$.

i. Show that $C_K(s, \omega, t) = C_K(s, X(s))$, i.e. it only depends on ω through $X(s)$.

ii. Show that $\frac{\partial^2 C_K(s,x,t)}{\partial K^2} = p(s, x, t, K)$ where p are the transition probabilities for $X(t)$.