PROBLEMS (due Mar 7)

1. The $n$th Hermite polynomial is $H_n(t, x) = \frac{(-1)^n}{n!} e^{\frac{t^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$. Show that the $H_n$ play the role that the monomials $\frac{x^n}{n!}$ play in ordinary calculus,
$$dH_{n+1}(t, B_t) = H_n(t, B_t) dB.$$

2. The backward equation for the Ornstein-Uhlenbeck process is
$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - \rho x \frac{\partial u}{\partial x}.$$ 
Show that $v(t, x) = u(t, xe^{\rho t})$ satisfies
$$e^{2\rho t} \frac{\partial v}{\partial t} = \frac{1}{2} \frac{\partial^2 v}{\partial x^2}$$
and transform this to the heat equation by $\tau = \frac{1-e^{-2\rho t}}{2\rho}$. Use this to derive Mehler’s formula for the transition probabilities,
$$p(t, x, dy) = e^{-\rho \frac{(y-xe^{\rho t})^2}{2(1-e^{-2\rho t})}} \frac{1}{\sqrt{2\pi \left( \frac{(1-e^{-2\rho t})}{2\rho} \right)^2}} dy.$$

3. i. Show that $X(t) = (1 - t) \int_0^t \frac{1}{1-s} dB(s)$ is the solution of
$$dX(t) = -\frac{X(t)}{1-t} dt + dB(t), \quad 0 \leq t < 1, \quad X(0) = 0.$$

ii. Show that $X(t)$ is Gaussian and find the mean and covariance.

iii. Show that for $0 = t_0 < t_1 < \cdots < t_n < 1$ the variables $\frac{X(t_i)}{1-t_i} - \frac{X(t_{i-1})}{1-t_{i-1}}$ are independent.

iv. Show that the finite dimensional distributions are given by
$$P(X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n)$$
$$= \prod_{i=1}^n p(t_i - t_{i-1}, x_i - x_{i-1}) \frac{p(1-t_n, -x_n)}{p(1,0)} dx_1 \cdots dx_n$$
where $p(t, x)$ is the Gaussian kernel.

v. Show that $X(t)$ is equal in distribution to a Brownian motion conditioned to have $B(1) = 0$. It is the Brownian Bridge.
vi. For fixed constants $a$ and $b$ solve the stochastic differential equation
\[
    dX(t) = \frac{b - X(t)}{1 - t} dt + dB(t), \quad 0 \leq t < 1 \quad X(0) = a.
\]
This is the Brownian Bridge from $a$ to $b$.

4. Consider the general linear stochastic differential equation
\[
    dX_t = [A(t)X_t + a(t)]dt + \sigma(t)dB_t, \quad X_0 = x,
\]
where $B_t$ is an $r$-dimensional Brownian motion independent of the initial vector $x \in \mathbb{R}^d$ and the $d \times d$, $d \times 1$ and $d \times r$ matrices $A(t)$, $a(t)$ and $\sigma(t)$ are non-random. Show that the solution is given by
\[
    X_t = \Phi(t)[x + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s]
\]
where $\Phi$ is the $d \times d$ matrix solution of $\dot{\Phi}(t) = A(t)\Phi(t)$, $\Phi(0) = I$.

5. If $P$ and $\tilde{P}$ are equivalent and $\frac{d\tilde{P}}{dP} = Z$ show that $\frac{dP}{d\tilde{P}} = \frac{1}{Z}$.
Let $P_{x}^{a,b}$ denote the probability measure on $C([0, T])$ corresponding to the solution of the stochastic differential equation
\[
    dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, \quad X(0) = x
\]
where $a = \sigma \sigma^T$. Let $b_1 \neq b_2$. Write expressions for $\frac{dP_{x}^{a,b_1}}{dP_{x}^{a,b_2}}$ and $\frac{dP_{x}^{a,b_2}}{dP_{x}^{a,b_1}}$ using the Cameron-Martin- Girsanov formula.
Is the second the inverse of the first, or not? Find an explanation.