PROBLEMS (due Mar 17)

1. Let $\tau_x = \inf\{t \geq 0 : B_t \geq x\}$. Prove that τ_x is equal in distribution to $x^2(\sup_{0\leq s\leq t} B_s)^{-1}$.

2. Let B^1_t and B^2_t be independent Brownian motions. Find the one dimensional marginal distribution of $\frac{B^1_t}{|B^2_t|}$ (the one dimensional marginal distribution of X_t means $F(t) = P(X_t \leq x)$, $t \geq 0$).

3. Let $B_t = (B^1_t, B^2_t)$ be a two dimensional Brownian motion starting at $(0,0)$. Let τ be the first time that B_t hits the line $\{(x,y) : y = a\}$. Compute the distribution of B_τ (Hint: Use problems 1 and 2.)

4. Let $\tau_{\pm x} = \inf\{t \geq 0 : |B_t| = x\}$. Compute the Laplace transform of $\tau_{\pm x}$.

5. Let G be a nice subset of the lattice $\mathbf{Z}^d$ and ∂G be its boundary. Let f be a function defined on the boundary. Let X_n be a symmetric nearest neighbour random walk on the lattice starting at $x \in G$ and let τ_G be the first time that the walk hits the boundary of G. Show that the solution of the discrete Dirichlet problem

$$\Delta u = 0 \quad \text{in } G, \qquad u = f \quad \text{on } \partial G$$

is given by $u(x) = E_x[f(X_{\tau_G})]$. The lattice Laplacian $\Delta u(x)$ is given by $\sum_{y\sim x} u(y) - u(x)$ where $y \sim x$ means y is a neighbour of x. (Note that the wording of the problem is purposefully vague and part of the problem is to make words like "subset", "nice" and "boundary" precise here.)

6. Let $\mathcal{O}$ be an orthogonal transformation of $L^2[0,\infty)$. Let $\mathbf{1}_{[0,t]}$ be the indicator function of $0 \leq s \leq t$. Define

$$\tilde{B}(t) = \int_0^\infty \left(\mathcal{O}\mathbf{1}_{[0,t]}\right)(s)dB(s).$$

Show that $B \mapsto \tilde{B}$ is a measure preserving transformation of the space of Brownian paths.

7. Let B_t, $t \geq 0$ be a d-dimensional Brownian motion. Show that the one dimensional marginal densities $f(t,x)$ satisfy the heat equation $\partial_t f = \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}$. Find the equation satisfied by the one dimensional marginals of the *Bessel process* $r_t = |B_t|$.

8. (a) Show that the solution $\mathbf{X}(t)$ of the stochastic differential equation

$$\begin{aligned} dX_1 &= -X_2 dB \\ dX_2 &= X_1 dB \end{aligned}$$

does *not* stay on the circle $|\mathbf{X}| = 1$, even though the vector field $(-X_2, X_1)$ is tangent to the circle at (X_1, X_2). (Hint: Differentiate $|X(t)|$.)

(b) Show that the solution $\mathbf{X}(t)$ of the stochastic differential equation

$$\begin{aligned} dX_1 &= -X_2 dB - \frac{1}{2} X_1 dt \\ dX_2 &= X_1 dB - \frac{1}{2} X_2 dt \end{aligned} \tag{1}$$

does stay on the circle.

(c) Show that the solution of (1) is given by $\mathbf{X}(t) = (\cos B(t), \sin B(t))$.

9. Let $\sigma(x)$ be a smooth, positive function. Solve the stochastic differential equation

$$dx = \frac{1}{2}\sigma'(x)\sigma(x)dt + \sigma(x)dB, \qquad x(0) = 0.$$

(Hint. Try $x = g(B)$ where g is the inverse function of the integral of $1/\sigma$.)

10. Let $x(t)$ be the solution of the stochastic differential equation $dx = \sigma(x)dB + b(x)dt$, $x(0) = x$ and let E_x denote expectation with respect to this process. Find a partial differential equation for

$$u(t, x) = E_x[e^{\int_0^t V(x(s))ds} u_0(x(t))].$$

(You can assume any reasonable conditions on $V(x)$, $\sigma(x)$, and b.)