PROBLEMS (due Feb 24)

1. Let $B(t)$ be Brownian motion. Prove that $X = \int_0^1 B^2(s)ds$ is a random variable and compute the first two moments. Is X Gaussian?

2. Let $B(t)$ be Brownian motion. Prove that with probability one,

$$\lim_{n\to\infty} \sum_{i<2^n t} |B_{(i+1)2^{-n}} - B_{i2^{-n}}|^2 = t$$

(Hint: Compute the variance and use Borel-Cantelli)

3. Let $p \in (-1/2, 1/2)$. For each fixed y prove that

$$f_y(x) = |x-y|^{-p} - |x|^{-p}$$

is in $L^2(\mathbf{R})$.

For a nice function f define $X(f) = \int_{-\infty}^{\infty} f dB = -\int_{-\infty}^{\infty} f'(s)B(s)ds$. Here $B(t)$, $t \in \mathbf{R}$ is a two sided Brownian motion, obtained as follows: Let $B_1(t)$ and $B_2(t)$ be independent Brownian motions starting at 0 and define $B(t) = B_1(t)$ for $t \geq 0$ and $B(t) = B_2(-t)$ for $t < 0$. Let

$$Z_y = X(f_y)$$

Z_y is called *fractional Brownian motion* of index α. Find the distribution (ie. all finite dimensional distributions) of Z_y.

4. Let τ be a stopping time. Show that

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \;:\; A \cap \{\tau \leq n\} \in \mathcal{F}_n,\; n \geq 0\}$$

is a σ-field.

5. If τ and σ are stopping times, then so are $\tau + \sigma$, $\max(\tau, \sigma)$ and $\min(\tau, \sigma)$.

6. Let B_t, $t \geq 0$ be Brownian motion. Use the law of the iterated logarithm

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t \log\log t^{-1}}} = 1 \qquad a.s.$$

to compute all possible limit points of

$$\frac{B_t}{\sqrt{t \log\log t}}$$

over subsequences $t_n \to \infty$.