PROBLEMS (due Jan 27)

1. Let $B(t)$, $0 \le t \le 1$ be Brownian motion. Show that
$$X(t) = B(1-t) - B(1)$$
is also Brownian motion.

2. (a) Let B_t, $t \ge 0$ be Brownian motion, f_1 and f_2 functions on $[0,1]$ and let $X_i = \int_0^1 f_i(t) B_t dt$. Find the distribution of $X = (X_1, X_2)$.

   (b) Compute the Fourier series of Brownian motion on $[0,1]$, ie. if $a_n = \int_0^1 e^{2\pi i n t} B_t dt$ compute all finite dimensional distributions of the sequence a_n.

3. If f is differentiable and non-random, define $X_t = \int_0^t f dB = f(t)B(t) - f(0)B(0) - \int_0^t f'(t)B(t)dt$.

   (a) Prove that X_t is continuous.

   (b) Compute $E[X_t]$ and $E[X_t^2]$.

4. Let $B(t)$, $t \ge 0$ be Brownian motion. Show that the density $f(t)$ of $B(t)$ satisfies the heat equation,
$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}.$$

5. Let $B_n(t)$ be the polygonalization of Brownian motion defined in the proof of continuity. Let $Y_n(t) = \frac{d}{dt} B_n(t)$

   (a) Show that $Y_n(t)$ has no limit as $n \to \infty$.

   (b) Compute the mean $m_n(t) = E[Y_n(t)]$ and covariance $\rho_n(t-s) = E[Y_n(t)Y_n(s)]$.

   (c) Compute the spectral density μ_n defined by $\rho_n(t) = \int_{-\infty}^{\infty} e^{it\lambda} d\mu_n(\lambda)$ and find the limit $\mu = \lim_{n\to\infty} \mu_n$.

   (d) Explain why the mythical $Y = \lim_{n\to\infty} Y_n$ is known as *white noise*.

6. A process X_t is said to be *stochastically continuous* at t_0 if for any $\epsilon > 0$
$$\lim_{t \to t_0} P(|X_t - X_{t_0}| > \epsilon) = 0.$$
Construct a process which is stochastically continuous at every point, but has discontinuities with probability one.

7. Let B_t, $t \ge 0$ be Brownian motion. For each n, let $t_i = i/n$, $i = 0, \ldots, n$. For each $p > 0$ prove that there are real numbers α_p, C_p such that in probability
$$\lim_{n\to\infty} n^{\alpha_p} \sum_{i=0}^{n} |B_{t_{i+1}} - B_{t_i}|^p = C_p.$$