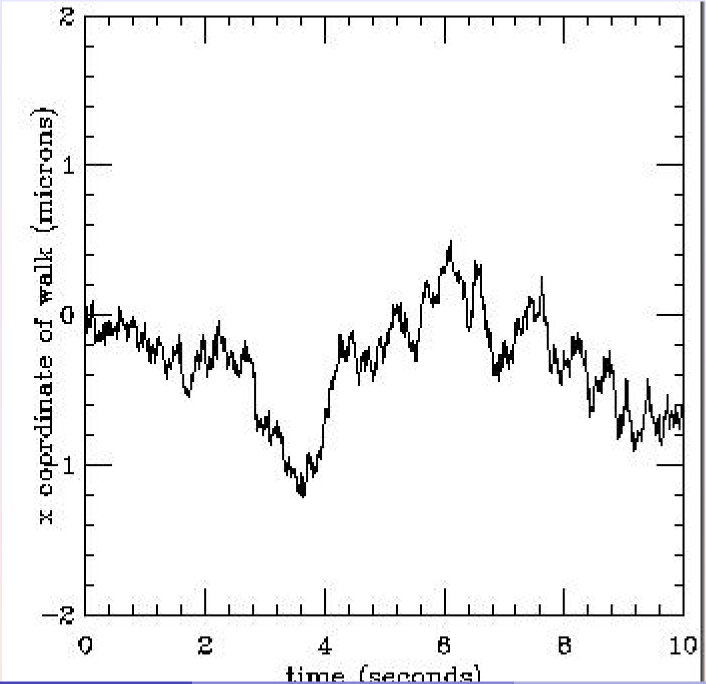

2
1
0
−1
−2
x coordinate of walk (microns)
0
2
4
6
8
10
time (seconds)

Outline

- Stochastic processes, Brownian motion, continuity
- Non-differentiabilty, Quadratic variation
- Conditional expectation, martingales, Markov property, stopping times
- Reflection principle, Dirichlet problem, recurrence/transience in $\mathbb{R}^d$
- Stochastic Integral, Ito's lemma, Ornstein-Uhlenbeck process,
- Local time, Feynman-Kac formula, Arcsin law
- Stochastic differential equations, existence/uniqueness, Bessel processes
- Backward and forward equations, applications
- Levy's theorem, time change, applications
- Cameron-Martin-Girsanov formula, Brownian bridge
- Martingale representation theorem, applications
- Diffusion limits of Markov chains

Random variables

A *probability space* $(\Omega, \mathcal{F}, P)$ consists of a set Ω, on which there is a σ*-field* $\mathcal{F}$

- if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
- if $A_i \in \mathcal{F}$, $i = 1, 2, 3, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

and a *probability measure*

- $0 = P(\emptyset) \leq P(A) \leq P(\Omega) = 1$, $A \in \mathcal{F}$
- if $A_i \in \mathcal{F}$, $i = 1, 2, 3, \ldots$ are disjoint, $A_i \cap A_j = \emptyset$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

A (real-valued) *random variable* is a measurable map

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

Distribution function $F(x) = P(X \leq x)$

Measure $\mu(A) = P(X \in A)$

Characteristic function $\varphi(t) = E[e^{itX}] = \int e^{itx} dF(x)$

Random variables in $\mathbb{R}^n$

n random variables $X_1, \ldots, X_n$

Can just think of it as a random n-vector $X = (X_1, \ldots, X_n)$ or a random element of $\mathbb{R}^n$

Distribution function

$$F(x_1, \ldots, x_n) = P(X_1 \leq x_1, \ldots, X_n \leq x_n)$$

Measure $\mu(A) = P(X \in A)$
Characteristic function $\varphi(t_1, \ldots, t_n) = E[e^{it \cdot X}]$.

Stochastic processes

A *stochastic process* is an indexed set of random variables

$$X_t, \quad t \in T$$

i.e. measurable maps from a probability space $(\Omega, \mathcal{F}, P)$ to a state space $(E, \mathcal{E})$ T = time

In this course $T = \mathbb{R}_+$ or $\mathbb{R}$ (continuous time)

But you could have $T = \mathbb{N}_+$ or $\mathbb{N}$ (discrete time).

In this course $E = \mathbb{R}$ or $\mathbb{R}^d$
$\mathcal{E} = \mathcal{B}(\mathbb{R}^d)$= the Borel σ-field
= the smallest σ-field containing all open sets

Fix $\omega \in \Omega$. $X_t(\omega)$ is a function of $t \in \mathbb{R}_+$.
So X_t is a random function.

Distribution of a stochastic process

Analogously: Discrete time stochastic process $X_1, X_2, \dots$ is a random sequence or a random element of $\prod_{i=1}^{\infty} \mathbb{R}$

Continuous time stochastic process $X_t,\ t \geq 0$ is a random function or a random element of $\Omega = \prod_{t \geq 0} \mathbb{R}$

Family of finite dim. distr.'s, one for each $t_1 < \cdots < t_n$,

$$F_{t_1,\dots,t_n}(x_1,\dots,x_n) = P(X_{t_1} \leq x_1,\dots,X_{t_n} \leq x_n) \tag{1}$$

Consistency: $F_{t_1,\dots,\hat{t}_i,\dots,t_n}(x_1,\dots,\hat{x}_i,\dots,x_n) = F_{t_1,\dots,t_n}(x_1,\dots,\infty,\dots,x_n)$

$\mathcal{F}$ generated by *cylinder sets* $E = \{X_{t_1},\dots,X_{t_n} \in A\}$, $A \in \mathcal{B}(\mathbb{R}^n)$

Kolmogorov Extension Theorem

Let $\{F_{t_1,\dots,t_n}\}_{t_1,\dots,t_n \in \mathbb{R}^n, n=1,2,\dots}$ *be a consistent family of finite dimensional distributions. There is a unique probability measure* P *on* $(\Omega, \mathcal{F})$ *satisfying (1).*

$X \sim \mathcal{N}(m, \sigma^2)$ if X is Gaussian with mean m and variance σ^2

$F(x) = \int_{-\infty}^{x} f(y)dy$, density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$ $\varphi(t) = e^{itm - \frac{1}{2}\sigma^2 t^2}$

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(m, C)$ if it has n-dimensional density

$$\frac{1}{\sqrt{2\pi \det C}} e^{-\frac{1}{2}(x-m)^T C^{-1}(x-m)}$$

If (X_1, X_2) is Gaussian, then they are independent if and only if they are orthogonal, i.e. $C_{12} = C_{21} = E[(X_1 - m_1)(X_2 - m_2)] = 0$.

Process is Gaussian if all finite dimensional distributions Gaussian

Theorem

Let $X_1, X_2, \ldots$ be independent and identically distributed with $E[X_i] = 0$ and $E[X_i^2] = \sigma^2 < \infty$. Then

$$\lim_{n\to\infty} P\left(\frac{X_1 + \cdots + X_n}{\sigma\sqrt{n}} \le x\right) = \int_{-\infty}^{x} \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} dy$$

Brownian motion

Brownian motion B_t, $t \geq 0$ is a Gaussian process with independent, stationary increments $B_{t+h} - B_t$

$B_{t_4} - B_{t_3}$, $B_{t_2} - B_{t_1}$ independent if $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$

Stationary: $B_{t+h} - B_t \overset{dist}{=} B_{s+h} - B_s$

$B_{t+h} - B_t \sim \mathcal{N}(0, h)$

$$P(B_{t_1} \leq x_1, \ldots, B_{t_n} \leq x_n) =$$

$$\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \frac{e^{-\frac{y_1^2}{2t_1}}}{\sqrt{2\pi t_1}} \frac{e^{-\frac{(y_2-y_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\frac{(y_n-y_{n-1})^2}{2(t_n-t_{n-1})}}}{\sqrt{2\pi(t_n-t_{n-1})}} dy_1 \cdots dy_n$$

Einstein's derivation (1905)

Particle starts at $0 \in \mathbf{R}^3$ and is pushed around by tiny molecular bombardments.

$$f(t,x) = \lim_{h \to 0} h^{-3} P(X_t \in \text{a box of side length h around x}).$$

$$p(s,x,t,y)dy = P(X_t \in dy \mid X_s = x).$$

$$f(t+\tau, x) = \int f(x-y,t)p(x-y,t,x,t+\tau)dy.$$

homogeneity in space and time $p(s,x,t,y) = p(t-s, y-x)$.

$$\begin{aligned} f(t+\tau,x) &= \int (f(t,x) - y\cdot\nabla f(t,x) + \frac{1}{2} y\cdot D^2 f(t,x) y + \cdots) p(\tau,y) dy \\ &= f(t,x)\int p(\tau,y)dy - \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(t,x)\int y_i p(\tau,y)dy \\ &+ \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\int y_i y_j p(\tau,y)dy + \cdots . \end{aligned}$$

$$\int p(\tau,y)dy = 1.$$

symmetry $\int y_i p(\tau,y)dy = 0$ and $\int y_i y_j p(\tau,y)dy = 0 \qquad i \neq j.$

Influence of molecular bombardment in any two nonoverlapping intervals of time is independent

Variance should grow linearly, like the sum of independent random variables

$$Var(X_1 + \cdots + X_N) \simeq CN$$

$$\int y_i^2 p(\tau, y) dy = D\tau.$$

Letting $\tau \to 0$ get heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} D \Delta f.$$

With the obvious initial condition $f(x, 0) = \delta_0$ this has the well known solution

$$f(t, x) = \frac{e^{-\frac{|x|^2}{2Dt}}}{(2\pi Dt)^{3/2}}.$$

Basic properties of Brownian motion

Mean: $m_t = E[B_t] = 0$

Covariance: If $s < t$,
$E[B_t B_s] = E[(B_t - B_s)B_s] + E[B_s^2] = E[B_t - B_s]E[B_s] + s = s$

B_t is the Gaussian process with

$$m_t = 0 \quad C_{s,t} = E[B_t B_s] = \min(s,t) \tag{2}$$

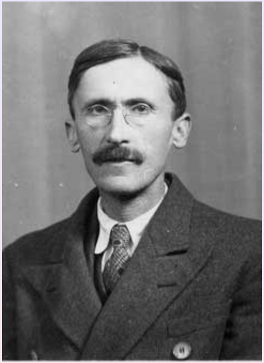
Proposition

Let B_t, $t \geq 0$ be a Brownian motion.

1. *For any $s \geq 0$, $\tilde{B}_t = B_{t+s} - B_s$, $t \geq 0$ is a Brownian motion independent of B_u, $u \leq s$*
2. *$-B_t$, $t \geq 0$ is a Brownian motion*
3. *For any a, $aB_{a^{-2}t}$, $t \geq 0$ is a Brownian motion*
4. *$tB_{1/t}$, $t \geq 0$ is a Brownian motion.*

Continuity of Brownian Motion

Norbert Wiener (1894-1964)



Paul Lévy (1886-1971)

Continuity of Brownian motion

We will construct Brownian motion directly on $\Omega = C([0,1])$, with its Borel σ-field, under the sup norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.

Lemma

If X is $\mathcal{N}(0,1)$ then $P(|X| \geq a) \leq e^{-a^2/2}$.

Proof.

We use the exponential Tchebyshev's inequality: For $\lambda > 0$,

$$P(|X| \geq a) \leq \frac{E[e^{\lambda X}]}{e^{\lambda a}}.$$

Recalling the Laplace transform of the normal distribution, the right hand side is $e^{-\lambda a + \lambda^2/2}$. To get the lemma, just choose $\lambda = a$. ☐

Continuity of Brownian motion: Haar functions

Definition

Haar basis of $L^2[0,1]$, $n = 0, 1, 2, \ldots$, $j = 0, \ldots, 2^n - 1$

$$\mathcal{H}_{j,n}(x) = 2^{\frac{n+1}{2}} \left\{ \mathbf{1}_{[\frac{2j}{2^{n+1}},\frac{2j+1}{2^{n+1}})} - \mathbf{1}_{[\frac{2j+1}{2^{n+1}},\frac{2j+2}{2^{n+1}})} \right\}$$

Integrals are Schauder functions $\mathcal{S}_{j,n}(x) = \int_0^x \mathcal{H}_{j,n}(y)dy$*, a tent of height* $2^{-\frac{n+1}{2}}$ *between* $\frac{2j}{2^{n+1}}$ *and* $\frac{2j+2}{2^{n+1}}$.

Let X, $\xi_{j,n}$ indep $\mathcal{N}(0,1)$ $\qquad B_0(t) = tX$,

$$B_{n+1}(t) = B_n(t) + \sum_{j=0}^{2^n-1} \xi_{j,n}\mathcal{S}_{j,n}(x) \qquad \xi_{j,m} \textit{ indep } \mathcal{N}(0,1)$$

Idea: $B_n(t) =$ polygonal approx. on the points $0, 1/2^n, 2/2^n, \ldots, 1$

$B_{n+1}(\frac{2j+1}{2^{m+1}}) - B_n(\frac{2j+1}{2^{m+1}}) = B_{n+1}(\frac{2j+1}{2^{m+1}}) - \frac{B_{n+1}(\frac{2j+2}{2^{m+1}}) + B_{n+1}(\frac{2j}{2^{m+1}})}{2}$

Proof that Brownian motion has continuous sample paths

$B_n(t,\omega)$ is continuous for each ω

Enough to show: $B_n(\omega)$ converges uniformly a.s.

i.e. $B(t)=\sum_{m=0}^{\infty} B_{m+1}(t)-B_m(t)+B_0(t)$ converges uniformly

$B_n(t)-B(t)=\sum_{m=n}^{\infty} B_{m+1}(t)-B_m(t)$

$$\sup_{0\le t\le 1}|B_n(t)-B(t)|\le\sum_{m=n}^{\infty}\sup_{0\le t\le 1}|B_{m+1}(t)-B_m(t)|$$

$$\sup_{0\le t\le 1}|B_{m+1}(t)-B_m(t)|=2^{-\frac{m+1}{2}}\max_{0\le j\le 2^m-1}|\xi_{j,m}|.$$

$$P(2^{-\frac{m+1}{2}}\max_{0\le j\le 2^m-1}|\xi_{j,m}|\ge 2^{-\frac{m}{4}})\le 2^m P(2^{-\frac{m+1}{2}}|\xi_{0,m}|\ge 2^{-\frac{m}{4}})\le 2^m e^{-2^{m/2}}$$

Continuity of Brownian motion: Proof continued

Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} P(A_n) < \infty$ *then almost every* ω *is in at most finitely many* A_n*.*

Proof of Borel-Cantelli lemma.

$$\{\omega : \omega \text{ in infinitely many } A_n\} = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$$

$$P(\cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n) = \lim_{m\to\infty} P(\cup_{n=m}^{\infty} A_n) \leq \lim_{m\to\infty} \sum_{n=m}^{\infty} P(A_n) = 0$$

□

Continuity of Brownian motion: Proof continued

Proof.

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big(\sum_{m=n}^{\infty} \sup_{0\le t\le 1} |B_{m+1}(t) - B_m(t)| \ge \sum_{m=n}^{\infty} 2^{-m/4}\Big) \\
&= \sum_{n=1}^{\infty} P\Big(\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0\le j\le 2^m-1} |\xi_{j,m}| \ge \sum_{m=n}^{\infty} 2^{-m/4}\Big) \\
&\le \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} 2^m e^{-2^{m/2}} < \infty
\end{aligned}$$

It follows that $\exists N_1(\omega)$ s.t. for $n \ge N_1(\omega)$

$$\sum_{m=n}^{\infty} \sup_{0\le t\le 1} |B_{m+1}(t) - B_m(t)| \le \sum_{m=n}^{\infty} 2^{-m/4}$$

$\forall \epsilon$, r.h.s.$\le \epsilon$ for all $n \ge N_2$. $\square$