Doob's inequality Let $X(t)$ be a right continuous submartingale with respect to $\mathcal{F}(t)$, $t \geq 0$

1. $P(\sup_{0\leq s\leq t} X(s) \geq \lambda) \leq \frac{1}{\lambda}\int_{\{\sup_{0\leq s\leq t} X(s)\geq\lambda\}} X^+(t)dP$
2. For $1 < p < \infty$, $E[\sup_{0\leq s\leq t} |X(s)|^p] \leq \left(\frac{p}{p-1}\right)^p E[|X(t)|^p]$

1 ⇒ 2: $P(\lambda) = P(\bar{X} \geq \lambda)$, $\bar{X} = sup_{0\leq s\leq t} X(s)$, $Y = X(t)$,

$$E[|\bar{X}|^p] = -\int_0^\infty \lambda^p dP(\lambda) = p\int_0^\infty \lambda^{p-1} P(\lambda)d\lambda$$

$$\leq p\int_0^\infty \lambda^{p-1}(\frac{1}{\lambda}\int_{\bar{X}>\lambda} YdP)d\lambda = p\int Y\int_0^{\bar{X}} \lambda^{p-2}d\lambda dP$$

$$= \frac{p}{p-1}\int Y\bar{X}^{p-1}dP \leq \frac{p}{p-1}(\int |Y|^p dP)^{1/p}(\int \bar{X}^p dP)^{1-\frac{1}{p}}$$

Existence and Uniqueness Theorem

$\sigma : \mathbb{R}^d \times [0,T] \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ be Borel measurable, $\exists A < \infty$,

$$\|\sigma(x,t)\| + |b(x,t)| \le A(1+|x|) \qquad x \in \mathbb{R}^d,\ 0 \le t \le T$$

and *Lipschitz*;

$$\|\sigma(x,t) - \sigma(y,t)\| + |b(x,t) - b(y,t)| \le A|x-y|.$$

$x_0 \in \mathbb{R}^d$ indep of B_t, $E[|x_0|^2] < \infty$.
Then there exists a unique solution X_t on $[0,T]$ to

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

and $E[\int_0^T |X_t|^2 dt] < \infty$.

Uniqueness means that if X_t^1 and X_t^2 are two solutions then

$$P(X_t^1 = X_t^2,\ 0 \le t \le T) = 1$$

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \Big\{ \partial_s f(s, X_s) + \mathcal{L}f(s, X_s) \Big\} ds \\ &\quad + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

$$\mathcal{L}f(t, x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{d} b_i(t, x) \frac{\partial f}{\partial x_i}(t, x)$$

$$a_{ij} = \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk} \qquad a = \sigma\sigma^T$$

Markov property

X_t can be obtained by solving the stochastic differential equation up to time $s < t$ and then solving in $[s, t]$ with initial condition X_s
By uniqueness this gives the same answer
Define the transition probability

$$p(s, x, t, A) = P(X_t^{s,x} \in A)$$

where $X_t^{s,x}$ is the solution starting at x at time s
From the construction we have

$$P(X_t^{0,x} \in A \mid \mathcal{F}_s) = p(s, X_s^{0,x}, t, A)$$

which is the Markov property

Diffusions

A diffusion is a Markov process with transition probabilities $p(s, x, t, dy)$ satisfying, for each $\delta > 0$ as $h \to 0$,

$$i. \qquad \frac{1}{h}\int_{|y-x|\geq\delta} p(t, x, t+h, dy) \to 0 \qquad \Rightarrow \text{ continuous paths}$$

$$ii. \qquad \frac{1}{h}\int_{|y-x|<\delta} (y-x)p(t, x, t+h, dy) \to b(t, x)$$

$$iii. \qquad \frac{1}{h}\int_{|y-x|<\delta} (y_i-x_i)(y_j-x_j)p(t, x, , t+h, dy) \to a_{ij}(t, x)$$

$$f(t,X_t) - f(0,X_0) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds = \int_0^t \nabla f(s,X_s)\cdot \sigma dB_s$$

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} \quad = \textit{generator}$$

$M_t = f(t,X_t) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds$ is a martingale

$$\begin{aligned} 0 &= E[f(t,X_t) - f(s,X_s) - \int_s^t \Big\{\partial_u + \mathcal{L}\Big\} f(u,X_u)du \mid \mathcal{F}_s] \\ &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \Big\{\partial_u + \mathcal{L}\Big\} f(u,y)p(s,x,u,y)dydu, \qquad X_s = x \end{aligned}$$

For any f,

$$\begin{aligned} 0 &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \left\{\partial_u + \mathcal{L}\right\} f(u,y)p(s,x,u,y)dydu \end{aligned}$$

Fokker-Planck (Forward) Equation

$$\begin{aligned} \frac{\partial}{\partial t} p(s,x,t,y) &= \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j} (a_{i,j}(t,y)p(s,x,t,y)) \\ &\quad - \sum_{i=1}^{d} \frac{\partial}{\partial y_i} (b_i(t,y)p(s,x,t,y)) \\ &= L_y^* p(s,x,t,y) \end{aligned}$$

$$\lim_{t \downarrow s} p(s,x,t,y) = \delta(y-x).$$

Kolmogorov (Backward) Equation

$$\begin{aligned} -\frac{\partial}{\partial s} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\frac{\partial^2 p(s,x,t,y)}{\partial x_i \partial x_j} \\ &\quad + \sum_{i=1}^{d} b_i(s,x)\frac{\partial p(s,x,t,y)}{\partial x_i} \\ &= L_x p(s,x,t,y) \end{aligned}$$

$$\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x).$$

Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

Forward $$\frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y), \quad t > s,$$

$$p(s,x,s,y) = \delta(y-x)$$

Backward $$-\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Formal derivation of the backward equation

$$p(s,x,t,A) = \int p(s,x,s+h,dy)p(s+h,y,t,A)$$

$$0 = \int p(s,x,s+h,dy)\Big\{p(s+h,y,t,A) - p(s,x,t,A)\Big\}$$

$$\begin{aligned}0 \;&=\; \int p(s,x,s+h,dy)\Big\{h\frac{\partial p(s,x,t,A)}{\partial s} + \sum_{i=1}^{d}(y_i - x_i)\frac{\partial p(s,x,t,A)}{\partial x_i} \\ &\qquad + \frac{1}{2}\sum_{i,j=1}^{d}(y_i - x_i)(y_j - x_j)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j} + \cdots\Big\}\end{aligned}$$

$$-\frac{\partial p(s,x,t,A)}{\partial s} = \sum_{i=1}^{d} b_i(t,x)\frac{\partial p(s,x,t,A)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j}$$

Real derivation

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \le s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time t

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,dz)$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$

$$u(s,x) = p(s,x,t,y) \quad \text{if} \quad -\frac{\partial}{\partial s}u = L_s u \quad 0 \le s < t \quad u(t,x) = \delta(x-y)$$

Existence result from PDE

Suppose that $a(t,x)$ and $b(t,x)$ are bounded and that there are $\alpha > 0$, $\gamma \in (0,1]$, $C < \infty$ such that for all $s, t \geq 0$, $x, y \in \mathbb{R}^d$,

i. $\quad \xi^T a(t,x)\xi \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d,$

ii. $\quad \|a(s,x) - a(t,y)\| + |b(s,x) - b(t,y)| \leq C(|x-y|^\gamma + |t-s|^\gamma).$

Then the backward equation has a solution and furthermore

$$p(s,x,t,A) = \int_A p(s,x,t,y)dy$$

with $p(s,x,t,y) \geq 0$ jointly continuous in s, x, t, y. Furthermore, $p(s,x,t,y)$ is the unique weak solution of the forward equation,i.e.

$$\int f(t,y)p(s,x,t,y)dy - f(s,x) = \int_s^t \int \{\partial_u + \mathcal{L}\} f(u,y)p(s,x,u,y)dydu$$

The solution X_t, $t \geq 0$ of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with $X_0 = x$ is a Markov process with infinitesimal generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \qquad a = \sigma\sigma^*.$$

Itô's formula

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \Big\{\partial_s f(s, X_s) + \mathcal{L} f(s, X_s)\Big\} ds \\ &\quad + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s)\frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

Under the previous conditions, the following are equivalent

1. $\exists B_t,\ t \geq 0,\ dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$
2. For each $\lambda \in \mathbf{R}^d$,

$$Z_\lambda(t) = e^{\lambda\{X_t - \int_0^t b(s,X_s)ds\} - \frac{1}{2}\int_0^t \lambda^T a(s,X_s)\lambda ds}$$

   is a martingale with respect to $\mathcal{F}_t$
3. For all smooth $f(t, x)$,

$$f(t, X_t) - \int_0^t \{\partial_s + L\} f(s, X_s)ds$$

   is a martingale with respect to $\mathcal{F}_t$
4. For all smooth $f(x)$,

$$f(X_t) - \int_0^t Lf(X_s)ds$$

   is a martingale with respect to $\mathcal{F}_t$

Brownian motion in $\mathbb{R}^d$

1. $B_t = (B_t^1, \ldots, B_t^d)$, B_t^i independent Brownian motions
2. B_t Markov with $P(B_t \in A \mid B_s = x) = \int_A \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|y-x|^2}{2(t-s)}} dy$
3. B_t has stationary independent mean zero increments with $E[|B_t - B_s|^2] = d(t-s)$
4. $e^{\lambda \cdot B_t - \frac{1}{2}|\lambda|^2 t}$ is a martingale for any λ

Note that 1 does not depend on the basis: If $B_t^1, \ldots, B_t^2$ independent and $\mathcal{O}$ is orthogonal, then the coordinates of $\mathcal{O}B_t$ are independent Brownian motions in fact

Theorem

Suppose X_1, X_2 independent and $\exists \theta \neq N\pi/2$ such that

$$X_1 \cos\theta + X_2 \sin\theta, \qquad -X_1 \sin\theta + X_2 \cos\theta \qquad \text{independent}$$

Then X_1, X_2 are Gaussians (Maxwell)

Dirichlet problem

Given a bounded open subset $G \subset \mathbf{R}^d$ and a continuous function $f : \partial G \to \mathbf{R}$ find a continuous function $u : \bar{G} \to R$ such that

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u|_{\partial G} = f \end{cases}$$

$$\Delta u \overset{def}{=} \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 2d \lim_{r \to 0} r^{-2} \left(\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS - u(x) \right)$$

Lemma

u harmonic in $G \Leftrightarrow u$ satisfies the mean value property: for all sufficiently small $r > 0$,

$$\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS = u(x)$$

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Proof.

Green's identity $\int_G v \Delta u dx = \int_G u \Delta v dx + \int_{\partial G} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \; dS$

$G = \{\delta < x < r\}, \qquad v = \begin{cases} \frac{\log r - \log |x|}{\log r - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - r^{2-d}}{\delta^{2-d} - r^{2-d}} & d > 2 \end{cases}$

let $\rho \downarrow 0$ □

B_t d-dimensional Brownian motion starting at $x \in G$

$$\tau_G = \inf\{t \geq 0 \ : \ B(t) \notin G\}$$

$$u(x) = E_x[f(B(\tau_G))]$$

"Theorem" If ∂G "nice" then u solves the Dirichlet problem

$$E_x[f(B(\tau_G))] = \int_{\partial G} f(y)\pi_G(x, dy), \quad \pi_G(x, \Gamma) = P_x(B(\tau_G) \in \Gamma), \quad \Gamma \subset \partial G$$

Example. $G = B(x, r), \quad \pi_G(x, \Gamma) = \frac{|\Gamma|}{|\partial S(x,r)|}, \qquad \Gamma \subset S(x, r)$

Brownian motion is invariant under rotations
$\therefore \pi_G(x, \cdot)$ is invariant under rotations

Proposition

G bounded open $\subset \mathbb{R}^d$, f bounded measurable on ∂G. Then $u(x) = E_x[f(B(\tau_G))]$ is harmonic in G.

Proof.

$B = B(x,r) \subset G \qquad \tau_B \leq \tau_G$

Strong Markov property: $\quad u(B(\tau_S)) = E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]$

$$\begin{aligned} u(x) = E_x[f(B(\tau_G))] &= E_x[E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]] \\ &= E_x[u(B(\tau_S))] \\ &= \int_{\partial S} u(y)\pi_S(x,dy) \\ &= \frac{1}{|\partial S|}\int_{\partial S} u(y)dS \end{aligned}$$

So u satisfies the mean value property in G. ☐

$$a \in \partial G$$

To complete the proof that u solves the Dirichlet problem we need

$$\lim_{x \to a,\ x \in G} E_x[f(B(\tau_G))] = f(a) \qquad \text{It is not always true!}$$

Proposition

If $\lim_{\substack{x \to a \\ x \in G}} P_x[\tau_G > \epsilon] = 0$, $\forall \epsilon > 0$ then for any bdd mble function $f : \partial G \to \mathbb{R}$ which is continuous at a, $\lim_{\substack{x \to a \\ x \in G}} E_x[f(B(\tau_G))] = f(a)$

Proof.

Need: $\lim_{x \to a,\ x \in G} P_x \left(|B(\tau_G) - x| < \delta\right) = 1$

$$\begin{aligned} P_x(|B(\tau_G) - x| < \delta) &\geq P_x(\sup_{0 \leq t \leq \epsilon} |B(t) - x| < \delta,\ \tau_G \leq \epsilon) \\ &\geq P_x(\sup_{0 \leq t \leq \epsilon} |B(t) - x| < \delta) - P_x(\tau_G \leq \epsilon) \\ &\to 1 \ \text{ as } \ x \to a, x \in G \ \text{ then } \ \epsilon \downarrow 0 \end{aligned}$$

Proposition

$a \in \partial G$ is *regular* if $P_a(\sigma_G = 0) = 1 \quad \sigma_G = \inf\{t > 0 \ : \ B(t) \notin G\}$
a regular $\Leftrightarrow \quad \lim_{x \to a,\ x \in G} E_x[f(B_{\tau_G})] = f(a)\ \forall f$ bdd mble, cont at a

Proof of $\Rightarrow$

Enough to prove $P_x(\sigma_G < \epsilon)$ lower semi-continuous in x
Then $\limsup_{\substack{x \to a \\ x \in G}} P_x(\sigma_G < \epsilon) \geq P_a(\sigma_G < \epsilon) = 1$ and $\sigma_G \geq \tau_G$
But $\int p(0, x, \delta, y) P_y(\exists s \in (0, \epsilon - \delta), B(s) \notin G)$ continuous
and $\uparrow P_x(\sigma_G < \epsilon)$ as $\delta \downarrow 0$

Examples

1. If ∂G is a smooth manifold near a then a is regular by LIL
2. If $\exists$ cone C of height $h > 0$ and vertex at a such that $C - \{a\} \subset \bar{G}^C$ then a is a regular (exterior cone condition)
3. $d \leq 2$ always,$d \geq 3$ $\exists$counterexamples