# Stochastic Integrals (Wiener)

If $f$ nice, deterministic

$$\int_0^t f(s)dB(s) = f(t)B(t) - f(0)B(0) - \int_0^t f'(s)B(s)ds$$

say $f(t) = 0$

$$\begin{aligned} E[(\int_0^t f(s)dB(s))^2] &= E[(\int_0^t f'(s)B(s)ds)^2] \\ &= \int_0^t \int_0^t f'(s_1)f'(s_2)\min(s_1, s_2)ds_1 ds_2 \\ &= -\int_0^t f'(s_2)\int_0^{s_2} f(s_1)ds_1 ds_2 \\ &= \int_0^t f^2(s)ds \end{aligned}$$

$$\int_0^t f(s)dB(s) \sim \mathcal{N}(0, \int_0^t f^2(s)ds)$$

# Ornstein-Uhlenbeck Process

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

is the solution of the *Langevin equation*

$$dX_t = -\alpha X_t dt + \sigma dB_t$$

The stochastic differential equation always means its integrated version

$$X_t - X_0 = -\alpha \int_0^t X_s ds + \sigma(B_t - B_0)$$

To check it

$$\int_0^t X_s ds = \int_0^t X_0 e^{-\alpha s} ds + \sigma \int_0^t \int_0^s e^{-\alpha(s-u)} dB_u ds$$

so we need

$$\int_0^t X_0 e^{-\alpha s} ds + \sigma \int_0^t \int_0^s e^{-\alpha(s-u)} dB_u ds = -\alpha^{-1}(X_t - X_0 - \sigma(B_t - B_0))$$

Proof:

$$
\begin{aligned}
&\int_0^t X_0 e^{-\alpha s} ds + \sigma \int_0^t \int_0^s e^{-\alpha(s-u)} dB_u ds \\
&= -\alpha^{-1} X_0 (e^{-\alpha t} - 1) + \int_0^t \int_u^t e^{-\alpha(s-u)} ds dB_u \\
&= -\alpha^{-1} X_0 (e^{-\alpha t} - 1) - \alpha^{-1} \int_0^t (e^{-\alpha(t-u)} - 1) dB_u \\
&= -\alpha^{-1} (X_t - X_0 - \sigma(B_t - B_0))
\end{aligned}
$$

So

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

solves

$$dX_t = -\alpha X_t dt + \sigma dB_t$$

Another way to check it
Call $F = \sigma dB$

$$dX = -\alpha X dt + F$$

$$de^{\alpha t} X = e^{\alpha t} F$$

$$e^{\alpha t} X_t - X_0 = \int_0^t e^{\alpha s} \sigma dB(s)$$

This only works because the equation is linear

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

If $X_0 \sim \mathcal{N}(m, V)$ indep of $B_t$, $t \geq 0 \Rightarrow X_t$ Gaussian process

$$m(t) = E[X_t] = me^{-\alpha t}$$

$$c(s,t) = \mathrm{Cov}(X_s, X_t) = [V + \frac{\sigma^2}{2\alpha}(e^{2\alpha \min(t,s)} - 1)]e^{-\alpha(t+s)}$$

*relaxation time* $= \alpha^{-1}$

$m = 0$, $V = \frac{\sigma^2}{2\alpha} \Rightarrow X_t$ stationary Gaussian $c(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha(t-s)}$, $s < t$

we say $e^{-\frac{x^2}{\sigma^2/\alpha}} \frac{1}{\sqrt{\pi\sigma^2/\alpha}}$ is *invariant measure* or *stationary measure*

$X_t$, $t \geq 0$ is Ornstein-Uhlenbeck process solving

$$dX_t = -\alpha X_t dt + \sigma dB_t \qquad X_0 \sim \mathcal{N}(0, \frac{\sigma^2}{2\alpha})$$

$Y_t = \int_0^t X_s ds$ "Physical" Brownian motion

$Y_t$ Gaussian mean 0

$$\begin{aligned} Cov(Y_s, Y_t) &= \int_0^t \int_0^s Cov(X_u, X_v) du dv \\ &= \sigma^2 \int_0^t \int_0^s \frac{e^{-\alpha|u-v|}}{2\alpha} du dv \\ &\to \min(t, s) \qquad as\ \alpha \to \infty\ if\ \sigma = \alpha \end{aligned}$$

# $\int_0^t B(s)dB(s)$

Based on experience with Riemann integrals

$$\int_0^t B(s)dB(s) = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor - 1} B(t_j^n)\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right)$$

for some choice of $t_j^n \in [\frac{j}{2^n}, \frac{j+1}{2^n}]$. Lets try two choices, the right and left endpoints.

$$L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor - 1} B(\frac{j}{2^n})\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right)$$

$$R_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor - 1} B(\frac{j+1}{2^n})\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right).$$

$$L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j}{2^n}) \left( B(\frac{j+1}{2^n}) - B(\frac{j}{2^n}) \right)$$

$$R_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j+1}{2^n}) \left( B(\frac{j+1}{2^n}) - B(\frac{j}{2^n}) \right).$$

$$R_t - L_t = t$$

$$R_t + L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} (B^2(\frac{j+1}{2^n}) - B^2(\frac{j}{2^n})) = B^2(t).$$

$$L_t = \frac{1}{2}([L_t + R_t] - [R_t - L_t]) = \frac{1}{2}(B^2(t) - t) \qquad R_t = \frac{1}{2}(B^2(t) + t)$$

The choice of $t_j^n$ matters! This is why Riemann told you to only integrate functions of bounded variation.

Which one is correct?

$\int_0^t BdB = \frac{1}{2}(B^2(t) - t)$ or $\frac{1}{2}(B^2(t) + t)$ ?

Or something else??

eg. midpoint rule gives $\int_0^t BdB = \frac{1}{2}B^2(t)$ which looks reasonable

Not really a mathematical question. A modeling question

Of all choices two have some special properties:

$L_t = \int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - t)$ is a martingale : Itô integral

Midpoint rule $\int_0^t B(s) \circ dB(s) = \frac{1}{2}B^2(t)$ looks like ordinary calculus :
Stratonovich integral

We will *always* use the Itô integral and think of Stratonovich as a simple transformation of it which is sometimes useful in applications

(eg. Math finance: Itô, Math biology: Sometimes Stratonovich)

## Definition: Progressively measurable

$\sigma(s, \omega)$ is called *progressively measurable* if

1. i. $\sigma(s, \omega)$ is $\mathcal{B}[0, \infty) \times \mathcal{F}$ measurable;
2. ii. For all $t \geq 0$, the map $[0, t] \times \Omega \to \mathbb{R}$ given by $\sigma(s, \omega)$ is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable.

$\mathcal{B}[0, t]$ denotes the Borel $\sigma$-algebra on $[0, t]$.
Informally, $\sigma(s, \omega)$ is *nonanticipating*= uses information about $\omega$ contained in $\mathcal{F}_s$.

## Definition: Simple Functions

$\sigma(s, \omega)$ is called *simple* if there exists a partition $0 \leq s_0 < s_1 < \cdots$ of $[0, \infty)$ and bounded random variables $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ such that $\sigma(s, \omega) = \sigma_j(\omega)$ for $s_j \leq s < s_{j+1}$.

## Definition: Stochastic Integral for Simple Functions

Given such a $\sigma(s,\omega) = \sigma_j(\omega)$ for $s_j \le s < s_{j+1}$, $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ define

$$\int_0^t \sigma(s,\omega)dB(s) = \sum_{j=0}^{J(t)-1} \sigma_j(\omega)(B(s_{j+1})-B(s_j))+\sigma_{J(t)}(\omega)(B(t)-B(s_{J(t)}))$$

where $s_{J(t)} < t \le s_{J(t)+1}$.

## Basic properties

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale

## Proof.

Since $\sigma_j \in \mathcal{F}_{s_j}$, if $u \ge s_j$, $E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_u] = \sigma_j(B(u) - B(s_j))$ and if $u < s_j$, $E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_u] =$ $E[E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_{s_j}] \mid \mathcal{F}_u] = 0$. □

## Basic properties

3 $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$

## Proof.

$\int_0^t \sigma dB = \sum_j \sigma_j(B(s_{j+1} \wedge t) - B(s_j))$

$E[(\int_0^t \sigma dB)^2] = \sum_{i,j} E[\sigma_i\sigma_j(B(s_{i+1} \wedge t) - B(s_i))(B(s_{j+1} \wedge t) - B(s_j))]$

$i < j : E[E[\sigma_i\sigma_j(B(s_{i+1} \wedge t) - B(s_i))(B(s_{j+1} \wedge t) - B(s_j)) \mid \mathcal{F}_{s_j}]] = 0$

$i = j : E[\sigma_j^2(B(s_{j+1} \wedge t) - B(s_j))^2] =$
$E[E[\sigma_j^2(B(s_{j+1} \wedge t) - B(s_j))^2 \mid \mathcal{F}_{s_j}] = E[\sigma_j^2](s_{j+1} \wedge t - s_j)$

$E[(\int_0^t \sigma dB)^2] = \sum_j E[\sigma_j^2(B(s_{j+1} \wedge t) - B(s_j))^2] = E[\int_0^t \sigma^2 ds]$

□

## Basic properties

4 $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale

## Proof.

Suppose $t \geq u \geq J(t)$. Then $E[e\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds \mid \mathcal{F}_u]$ can be written

$$e^{\sum_{j=0}^{J(t)-1} \sigma_j(B(s_{j+1})-B(s_j))-\frac{1}{2}\sigma_j^2(s_{j+1}-s_j)} E[e^{\sigma_{J(t)}(B(t)-B(s_{J(t)})-\frac{1}{2}\sigma_{J(t)}^2(t-s_{J(t)})} \mid \mathcal{F}_u].$$

The expectation is just 1 , so we have that $E[Z(t) \mid \mathcal{F}_u] = Z(u)$ whenever $t \geq u \geq J(t)$. It follows by repeated conditioning that $E[Z(t) \mid \mathcal{F}_u] = Z(u)$ for any $u \leq t$. □

$\mathcal{P}$ = set of progressively measurable functions

## Lemma

For each $t.0$, $\mathcal{P}$ = Closure in $L^2([0,t] \times \Omega, dt \times dP)$ of simple functions

## Proof

Suppose $\sigma \in \mathcal{P}$ and $E[\int_0^t \sigma^2(s,\omega)ds] < \infty$
we need to find a sequence $\sigma_n$ of simple functions s.t.

$$E[\int_0^t (\sigma(s,\omega) - \sigma_n(s,\omega))^2 ds] \to 0.$$

We can assume that $\sigma$ is bounded For if $\sigma_N = \sigma$ for $|\sigma| \leq N$ and 0 otherwise then $\sigma_N \to \sigma$ and $|\sigma_N - \sigma|^2 \leq 4|\sigma|^2$ so by the dominated convergence theorem $E[\int_0^t (\sigma - \sigma_N)^2 ds] \to 0$.

## Proof.

Furthermore we can assume that $\sigma$ is continuous in $s$

for if $\sigma$ is bounded then $\sigma_h = h^{-1}\int_{t-h}^{t}\sigma ds$ are continuous progressively measurable and converge to $\sigma$ as $h \to 0$. By the bounded convergence theorem

$$E[\int_0^t (\sigma - \sigma_h)^2 ds] \to 0$$

For $\sigma$ continuous bounded and progressively measurable let

$$\sigma_n(s,\omega) = \sigma(\frac{\lfloor ns \rfloor}{n}, \omega)$$

These are progressively measurable, bounded and simple functions converging to $\sigma$ and again by the bounded convergence theorem,

$$E[\int_0^t (\sigma - \sigma_n)^2 ds] \to 0$$

□

## Theorem (Definition of the Itô Integral)

Let $\sigma(s,\omega)$ be progressively measurable and for each $t \geq 0$, $E[\int_0^t \sigma^2 ds] < \infty$. Let $\sigma_n$ be simple functions with $E[\int_0^t (\sigma_n - \sigma)^2 ds] \to 0$ and set

$$X_n(t,\omega) = \int_0^t \sigma_n(s,\omega) dB(s).$$

Then

$$X(t,\omega) = \lim_{n\to\infty} X_n(t,\omega)$$

exists uniformly in probability, i.e. for each $T > 0$ and $\epsilon > 0$,

$$\lim_{n\to\infty} P(\sup_{0\leq t\leq T} |X_N(t,\omega) - X(t,\omega)| \geq \epsilon) = 0.$$

Furthermore the limit is independent of the choice of approximating sequence $\sigma_n \to \sigma$. The limit $X(t,\omega)$ is the Itô integral

$$X(t) = \int_0^t \sigma(s) dB(s)$$

## Proof.

$X_n(t) - X_m(t) = \int_0^t (\sigma_n - \sigma_m) dB$ is a continuous martingale so by Doob's inequality

$$\begin{aligned} P(\sup_{0 \le t \le T} |X_n(t) - X_m(t)| \ge \epsilon) &\le \epsilon^{-2} E[(X_n - X_m)^2(T)] \\ &= \epsilon^{-2} E[\int_0^T (\sigma_n - \sigma_m)^2 ds] \end{aligned}$$

So $X_n - X_m$ is uniformly Cauchy in probability and therefore there exists a progressively measurable $X$ with

$$P(\sup_{0 \le t \le T} |X(t,\omega) - X_n(t,\omega)| \ge \epsilon) \stackrel{n\to\infty}{\to} 0 \qquad \epsilon > 0$$

If $\sigma'_n \stackrel{L^2}{\to} \sigma$ and $X'_n = \int_0^t \sigma'_n dB$, $P(\sup_{0 \le t \le T} |X_n - X'_n| \ge \epsilon) \to 0$ so that $X_n$ and $X'_n$ have the same limit. □

## Basic properties of the Itô Integral

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale.
3. $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$.
4. If $|\sigma| \leq C$ then $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale

## proof

1. By construction
2. Continuity follows from the construction. To prove the limit is a martingale we have $E[X_n(t) \mid \mathcal{F}_s] = X_n(s)$ and $X_n \to X$ in $L^2$, therefore in $L^1$ as well. The $L^1$ limit of a martingale is a martingale.
3. $X_n^2(t) - \int_0^t \sigma_n^2(s)ds$ is a martingale $\xrightarrow{L^1} X^2(t) - \int_0^t \sigma^2(s)ds$
4. $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$ is a martingale so it suffices to show that $Z_n(t)$, $n = 1, 2, \ldots$ is a uniformly integrable family.

## Proof.

to show that $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$, $n = 1, 2, \ldots$ is a uniformly integrable family, it is enough to show that there is some fixed $C < \infty$ for which $E[(Z_N(t))^2] \le C$.

$$\begin{aligned} E[(Z_N(t))^2] &= E[\exp\{2\int_0^t \sigma_n dB - \int_0^t \sigma_n^2 ds\}] \\ &\le e^{Ct} E[\exp\{2\int_0^t \sigma_n dB - \frac{4}{2}\int_0^t \sigma_n^2 ds\}] \\ &= e^{Ct} \end{aligned}$$

□

A *stochastic integral* is an expression of the form

$$X(t,\omega) = \int_0^t \sigma(s,\omega)dB(s) + \int_0^t b(s,\omega)ds + X_0$$

where $\sigma$ and $b$ are progressively measurable with $E[\int_0^t \sigma^2(s,\omega)ds] < \infty$ and $\int_0^t |b(s,\omega)|ds < \infty$ for all $t \geq 0$, and $X_0 \in \mathcal{F}_0$ is the starting point

The stochastic differential

$$dX = \sigma dB + bdt$$

is shorthand for the same thing

For example the integral formula $\int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - t)$ can be written in differential notation as

$$dB^2 = 2BdB + dt$$

What happens if $B^2(t)$ is replaced by a more general function $f(B(t))$ ?

## Itô's Lemma

Let $f(x)$ be twice continuously differentiable. Then

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt$$

## Proof

First of all we can assume without loss of generality that $f$, $f'$ and $f''$ are all uniformly bounded , for if we can establish the lemma in the uniformly bounded case, we can approximate $f$ by $f_n$ so that all the corresponding derivatives are bounded and converge to those of $f$ uniformly on compact sets.

Let $s = t_0 < t_1 < t_2 < \cdots < t_n = t$. We have

$$\begin{aligned} f(B(t)) - f(B(s)) &= \sum_{j=0}^{n-1} [f(B(t_{j+1})) - f(B(t_j))] \\ &= \sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \\ \xi_j \in [t_j, t_{j+1}] \qquad & \quad + \sum_{j=0}^{n-1} \frac{1}{2} f''(B(\xi_j))(B(t_{j+1}) - B(t_j))^2, \end{aligned}$$

Let the width of the partition go to zero. By definition of the stochastic integral

$$\sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_s^t f' dB.$$

Finally we want to show that

$$\sum_{j=0}^{n-1} f''(B(\xi_j))(B(t_{j+1}) - B(t_j))^2 \to \int_s^t f''(B(u))dB(u)$$

It's a lot like the computation of the quadratic variation.
Suppose we had $f''(B(t_j)$ inside instead of $f''(B(\xi_j))$

$$E\left[\left(\sum_{j=0}^{n-1} f''(B(t_j))\left[(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\right]\right)^2\right]$$

$$= \sum_{i,j=0}^{n-1} E\left[f''(B(t_i))X_i f''(B(t_j))X_j\right] \qquad X_j = (B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)$$

Suppose $i < j$

$$\begin{aligned} E\left[f''(B(t_i))X_i f''(B(t_j))X_j\right] &= E\left[E\left[f''(B(t_i))X_i f''(B(t_j))X_j \mid \mathcal{F}(t_j)\right]\right] \\ &= E\left[f''(B(t_i))X_i f''(B(t_j))E\left[X_j \mid \mathcal{F}(t_j)\right]\right] = 0 \end{aligned}$$

Hence

$$E\left[\left(\sum_{j=0}^{n-1} f''(B(t_j))\left[(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\right]\right)^2\right]$$

$$= \sum_{j=0}^{n-1} E\left[(f''(B(t_i))^2\left[(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\right]^2\right] \to 0$$

so

$$\sum_{j=0}^{n-1} f''(B(t_j))(B(t_{j+1}) - B(t_j))^2 \xrightarrow{L^2} \int_s^t f''(B(u))du$$

We still have to show $\sum_{j=0}^{n-1} |f''(B(\xi_j)) - f''(B(t_j)|(B(t_{j+1}) - B(t_j))^2 \to 0$
Taking expectation we get

$$\begin{aligned}
&\sum_{j=0}^{n-1} E[|f''(B(\xi_j)) - f''(B(t_j)|(B(t_{j+1}) - B(t_j))^2] \\
&\leq \sum_{j=0}^{n-1} \sqrt{E[(f''(B(\xi_j)) - f''(B(t_j))^2]}\sqrt{E[(B(t_{j+1}) - B(t_j))^4]} \\
&= C\sum_{j=0}^{n-1} \sqrt{E[(f''(B(\xi_j)) - f''(B(t_j))^2]}(t_{j+1} - t_j) \to 0
\end{aligned}$$

by continuity
So we have proved that

$$f(B(t) - f(B(s)) = \int_s^t f'(B(u))dB(u) + \frac{1}{2}\int_s^t f''(B(u))du$$

which is Itô's formula.

1. In differential notation Itô's formula reads

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt.$$

   The Taylor series is $df(B) = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(B)(dB)^n$. In normal calculus we would have $(dB)^n = 0$ if $n \geq 2$, but because of the finite quadratic variation of Brownian paths we have $(dB)^2 = dt$, while still $(dB)^n = 0$ if $n \geq 3$.

2. If the function $f$ depends on $t$ as well as $B(t)$, the formula is

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \frac{\partial f}{\partial x}(t, B(t))dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B(t))dt.$$

   The proof is about the same as the special case above.

# Local time

$f$ continuous function on $\mathbb{R}_+$

$$L_t(x) = \int_0^t \delta_x(f(s))ds = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} |\{0 \le s \le t : |f(s) - x| \le \epsilon\}|$$

$$\int_0^t 1_A(f(s))ds = \int_A L_t(x)dx$$

$f \in C^1$ $L_t(x) = \sum_{s_i \in [0,t]: f(s_i) = x} |f'(s_i)|^{-1}$ discontinuous in $t$
Itô 's lemma applied to $|B_t - x|$ gives

## Tanaka's formula for Brownian Local Time

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \operatorname{sgn}(B_s - x)dB_s$$

In particular, $L_t(x)$ continuous in $t$ a.s.

But $|x|$ not bounded, so no fair!!!

**Proof.**

$$f_\epsilon''(x) = (2\epsilon)^{-1}1_{[-\epsilon,\epsilon]}$$

Itô

$$(2\epsilon)^{-1}|\{0 \le s \le t : |B_s| \le \epsilon\}| = f_\epsilon(B_t) - f_\epsilon(B_0) - \int_0^t f_\epsilon'(B_s)dB_s$$

$$\epsilon \downarrow 0$$

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \mathrm{sgn}(B_s - x)dB_s$$

□