Martingales: Discrete time

Definition.

An non-decreasing family of sub-σ-fields $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ is called a *filtration*

Definition

M_n a sequence of random variables in $L^1(\Omega, \mathcal{F}, P)$. If

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n$$

then M_n is a *martingale* with respect to the filtration $\mathcal{F}_n$

submartingale: $E[M_{n+1} \mid \mathcal{F}_n] \geq M_n$

supermartingale: $E[M_{n+1} \mid \mathcal{F}_n] \leq M_n$

Example. $S_n = X_1 + \cdots + X_n$, X_i iid

$E[X_i] = 0 \Rightarrow S_n$ martingale. $E[X_i] \geq 0 \Rightarrow S_n$ submartingale.
$E[X_i] \leq 0 \Rightarrow S_n$ supermartingale

Doob's inequality (Discrete time)

Let X_n be a submartingale with respect to $\mathcal{F}_n$. Then for any $\lambda > 0$ and $n = 1, 2, \ldots,$

$$P\left(\max_{1\le k\le n} X_k \ge \lambda\right) \le \frac{E[X_n^+]}{\lambda}.$$

Proof.

$A_i = \left\{X_i \ge \lambda, \max_{0\le k\le i-1} X_k < \lambda\right\}$ disjt $\cup_{i=1}^n A_i = \{\max_{1\le i\le n} X_i \ge \lambda\}$

$$P\left(\max_{1\le i\le n} X_i \ge \lambda\right) = \sum_{i=1}^n P(A_i) \le \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} X_i dP \qquad \textit{Tchebyshev}$$

$$\le \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} E[X_n \mid \mathcal{F}_i] dP = \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} X_n dP$$

$$= \frac{1}{\lambda}\int_{\{\max_{1\le i\le n} X_i \ge \lambda\}} X_n dP$$

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$ a filtration A random variable τ taking values in $\{0, 1, 2, \ldots\}$ is called a *stopping time* if for each $n = 0, 1, 2, \ldots$,

$$\{\omega \in \Omega \;:\; \tau(\omega) \leq n\} \in \mathcal{F}_n.$$

Example

Let X_n be a random walk starting at 0. Let $\tau = \min\{n \geq 0 \;:\; X_n \geq a\}$ be the *first passage time* of level a. τ is a stopping time.

Let $\sigma = \max\{n \geq 0 \;:\; X_n \leq a\}$, the *last passage time*. σ is not a stopping time.

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \;:\; A \cap \{\tau \leq n\} \in \mathcal{F}_n,\; n \geq 0\}$$

is a σ-field representing the information up to the stopping time τ

Optional stopping

M_n martingale wrt filtration $\mathcal{F}_n$. $\tau \geq \sigma$ bounded stopping times

$$E[M_\tau \mid \mathcal{F}_\sigma] = M_\sigma$$

bounded means $\tau \leq B$ Otherwise it is FALSE

Proof.

Need: $\int_A MX_\tau dP = \int_A M_\sigma dP, \quad \forall\ A \in \mathcal{F}_\sigma$
$\int_{A\cap\{\sigma=\ell\}} MX_\sigma dP = \int_{A\cap\{\sigma=\ell\}} M_B dP$ since $A \cap \{\sigma = \ell\} \in \mathcal{F}_\ell$
so $\int_A M_\sigma dP = \int_A M_B dP$ same for M_τ since $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$

□

Example

$X_1, X_2, \ldots$ iid $P(X_i = 1) = P(X_i = -1) = 1/2$
$S_n = X_1 + \cdots + X_n$ Random walk
$\tau_{\pm a} = \min\{n : |S_n| = a\} \quad E[\tau_{\pm a}] = ?$
$S_n^2 - n$ martingale
$\tau_{\pm a}^B = \min\{\tau_{\pm a}, B\}$ bounded stopping time
Optional stopping: $E[S^2_{\tau^B_{\pm a}} - \tau^B_{\pm a}] = 0$
$\lim_{B\uparrow\infty} E[\tau^B_{\pm a}] = E[\tau_{\pm a}]$ by monotone convergence theorem
$\lim_{B\uparrow\infty} E[S^2_{\tau^B_{\pm a}}] = a^2$ by bounded convergence theorem
$E[\tau_{\pm a}] = a^2$

Counterexample

Try same for $\tau_a = \min\{n : S_n = a\}$
$\lim_{B\uparrow\infty} E[S^2_{\tau^B_a}] = E[\tau_a]$
but $\lim_{B\uparrow\infty} E[S^2_{\tau^B_a}] = \infty \neq E[S^2_{\tau_a}] = a^2$

Counterexample

$\tau_a = \min\{n : S_n = a\}$

$M_n = e^{\lambda S_n - n\psi(\lambda)} \qquad \psi(\lambda) = \log \frac{e^{\lambda}+e^{-\lambda}}{2}$

$E[e^{\lambda S_{\tau_a^B} - \tau_a^B \psi(\lambda}] = 1$

$\lambda > 0$ can take limit inside expectation by BCT

$E[e^{\lambda S_{\tau_a} - \tau_a \psi(\lambda)}] = 1$

$E[e^{-\tau_a \psi(\lambda)}] = e^{-\lambda a}$

$E[\tau_a] = \frac{1}{\psi'(0)} = \infty$

Martingales: Continuous time

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space

$\mathcal{F}_t$, $t \geq 0$ a *filtration* ($=$ non-decreasing family of sub-σ-fields of $\mathcal{F}$)

M_t, $t \geq 0 \in L^1$ is a *martingale* with respect to $\mathcal{F}_t$, $t \geq 0$ if whenever $s \leq t$,

$$E[M_t \mid \mathcal{F}_s] = M_s.$$

submartingale if $\geq$ *supermartingale* if $\leq$

Examples

- B_t is a martingale wrt $\mathcal{F}_t = \sigma(B_s, s \leq t)$
- B_t^2 is a submartingale
- $B_t^2 - t$ is a martingale
- $e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale for any $\lambda \in \mathbb{R}$

Martingale characterization of Brownian motion

If $e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale wrt $\mathcal{F}_t = \sigma(B_s, s \leq t)$ for any $\lambda \in \mathbb{R}$ then B_t, $t \geq 0$ is Brownian motion

Proof.

$$E[e^{\lambda(B_t - B_s)}|\mathcal{F}_s] = e^{\frac{1}{2}\lambda^2(t-s)}$$

so $B_t - B_s$ independent of $\mathcal{F}_s$ and $\mathcal{N}(0, t-s)$

□

Doob's inequality

If X_t is a submartingale with respect to $\mathcal{F}_t$ and the paths of X_t are right continuous with probability one, then

$$P(\sup_{0\leq t\leq T} X_t \geq \lambda) \leq \frac{E[X_T^+]}{\lambda}$$

Proof.

Let $0 \leq t_0 < t_1 < \cdots$ $\tilde{X}_n = X_{t_n}$ is a martingale wrt $\tilde{\mathcal{F}}_n = \mathcal{F}_{t_n}$.

$$P(\sup_{0\leq t_i\leq T} X_{t_i} \geq \lambda) \leq \frac{E[X_T^+]}{\lambda}$$

By right continuity lhs $\uparrow P(\sup_{0\leq t\leq T} X_t \geq \lambda)$ as mesh$\downarrow 0$ □

Optional stopping

X_t, $t \geq 0$ be a right continuous martingale with respect to $\mathcal{F}_t$, $t \geq 0$ and $\sigma \leq \tau$ bounded stopping times

$$E[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma$$

Proof.

$\sigma_n = 2^{-n}(\lfloor 2^n \sigma \rfloor + 1)$
$\tau_n = 2^{-n}(\lfloor 2^n \tau \rfloor + 1)$
$\sigma_n \leq \tau_n \leq B$
$E[X_{\tau_n} \mid \mathcal{F}_{\sigma_n}] = X_{\sigma_n}$ ie $\int_A X_{\tau_n} dP = \int_A X_{\sigma_n} dP$, $A \in \mathcal{F}_\sigma$,since $\sigma \leq \sigma_n$
By right continuity $X_{\tau_n} \to X_\tau$ and $X_{\sigma_n} \to X_\sigma$

Recall $\{X_n\}_{n=1,2,\ldots}$ is *uniformly integrable* if

$$\lim_{M \uparrow \infty} \sup_n \int_{|X_n| \geq M} |X_n| dP = 0$$

and if $X_n \xrightarrow{a.s.} X$ then $\{X_n\}_{n=1,2,\ldots}$ uniformly integrable $\Leftrightarrow X_n \xrightarrow{L^1} X$

X_{σ_n}, $n = 1, 2, \ldots$ and X_{τ_n}, $n = 1, 2, \ldots$ are *backwards martingales* with respect to $\mathcal{F}_n$, $n = 1, 2, \ldots$, i.e. $E[X_{\sigma_{n-1}} \mid \mathcal{F}_{\sigma_n}] = X_{\sigma_n}$

Lemma

A backwards martingale is uniformly integrable

Proof.

$E[X_m \mid \mathcal{F}_n] = X_n$ whenever $m \leq n$ so $|X_n| \leq E[|X_0| \mid \mathcal{F}_n]$
so

$$\int_{\{|X_n|>\ell\}} |X_n| dP \leq \int_{\{|X_n|>\ell\}} |X_0| dP = \int 1_{\{|X_n|>\ell\}} |X_0| dP$$

$$P(|X_n| > \ell) \leq \frac{E[|X_n|]}{\ell} \leq \frac{E[|X_0|]}{\ell}$$

so $1_{\{|X_n|>\ell\}} |X_0| \stackrel{a.s.}{\to} 0$

$\int_{\{|X_n|>\ell\}} |X_0| dP \to 0$ by dominated convergence theorem □

Strong Markov property of Brownian motion

B_t, $t \geq 0$ be a Brownian motion with respect to $\mathcal{F}_t$, $t \geq 0$
τ a bounded stopping time.
$\tilde{B}_t = B_{t+\tau} - B_\tau$.
Then $\tilde{B}_t$ is a Brownian motion independent of $\mathcal{F}_\tau$.

In other words, Brownian motion starts afresh at every stopping time.

Proof.

optional stopping $+$ martingale characterization of Brownian motion □

Reflection Principle

For all $x > 0$, $t \geq 0$,

$$P(\sup_{0 \leq s \leq t} B_s \geq x) = 2P(B_t \geq x)$$

In particular $\sup_{0 \leq s \leq t} B_s \overset{d}{=} |B_t|$

Proof.

$\tau_x = \inf\{t \geq 0 : B_t \geq x\}$

$$P(\sup_{0 \leq s \leq t} B_s \geq x) = P(B_t \geq x) + P(\sup_{0 \leq s \leq t} B_s \geq x, B_t < x) = P(\tau_x \leq t)$$

$$P(\sup_{0 \leq s \leq t} B_s \geq x, B_t < x) = P(\tau_x \leq t, B_{\tau_x + (t - \tau_x)} - B_{\tau_x} < 0) = \frac{1}{2} P(\tau_x \leq t)$$

□

$$P(\tau_x \leq t) = 2\int_x^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$$

Differentiate in t to see that τ_x has density

$$\begin{aligned} f_x(t) &= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{t^{3/2}} \int_x^{\infty} e^{-\frac{y^2}{2t}} dy + \frac{1}{t^{5/2}} \int_x^{\infty} y^2 e^{-\frac{y^2}{2t}} dy \right) \\ &= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \end{aligned}$$

In particular $E[\tau_x] = \infty$

$$\tau_x = \inf\{t \geq 0 : x^{-1} B_t = 1\} = \inf\{x^2 t : \tilde{B}_t = x^{-1} B_{x^2 t} = 1\} = x^2 \tilde{\tau}_1$$

Note τ_x has stationary, independent increments, i.e. it is a Lévy process

Another way to compute the distribution of τ_x, $x \geq 0$

$M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale

Optional stopping $\Rightarrow E[e^{\lambda B_{\min(\tau_x,B)} - \frac{1}{2}\lambda^2 \min(\tau_x,B)}] = 1$

BCT $\Rightarrow E[e^{\lambda B_{\tau_x} - \frac{1}{2}\lambda^2 \tau_x}] = 1, \qquad \lambda \geq 0$

so $E[e^{-\lambda \tau_x}] = e^{-\sqrt{2\lambda} x}, \qquad \lambda \geq 0$

In particular if τ_x and $\tilde{\tau}_y$ are independent then $\tau_x + \tilde{\tau}_y \stackrel{dist}{=} \tau_{x+y}$

Stable laws

X has stable distribution if for each n there is $0 < \alpha \leq 2$ and μ_n such that if $X_1, X_2, \ldots$ are iid with $X_i \stackrel{dist}{=} X$ then

$$\frac{X_1 + \cdots + X_n - \mu_n}{n^{1/\alpha}} \stackrel{dist}{=} X$$

Examples: Gaussian $\quad \alpha = 2$; Cauchy $\quad \alpha = 1$; $\tau_1 \quad \alpha = 1/2$

Subordinator means X_t non-decreasing (so it can be used as a time).

τ_x, $x \geq 0$ is a stable subordinator eg. $B_{\tau_x} = x$

$\tau = \inf\{t \geq 0 : B_t \textit{ hits } b + mt\}$, $b, m > 0$. What is $P(\tau < \infty)$?

$$E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)}] = 1$$

$B_{\min(\tau,N)} \leq b + m\min(\tau, N)$ so if $\lambda m \leq \frac{1}{2}\lambda^2$ we can let $N \to \infty$ to get

$$E[e^{\lambda(b+m\tau) - \frac{1}{2}\lambda^2\tau} 1_{\tau<\infty}] + \lim_{N\to\infty} E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] = 1$$

$$\begin{aligned} &\lim_{N\to\infty} E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] \\ &\leq \lim_{N\to\infty} E[e^{\lambda(b+m\min(\tau,N)) - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] \\ &= 0 \end{aligned}$$

so

$$E[e^{(\lambda m - \frac{1}{2}\lambda^2)\tau} 1_{\tau<\infty}] = e^{-\lambda b}$$

Let $\lambda \downarrow 2m$ to get

$$P(\tau < \infty) = e^{-2mb}$$

Brownian motion starting at x: $B_t + x$, $B_0 = 0$

$P_x(A) = P(A \mid B_0 = x)$

$a < x < b$ What is $P_x(\tau_a < \tau_b)$?

If $B_0 = x$ then B_t is still a martingale

$E_x[B_{\min(\tau_a,\tau_b,N)}] = x$

$N \to \infty$, BCT$\Rightarrow$ $E_x[B_{\min(\tau_a,\tau_b)}] = x = E_x[B_{\tau_a} 1_{\tau_a<\tau_b}] + E_x[B_{\tau_b} 1_{\tau_a>\tau_b}]$
$= aP_x(\tau_a < \tau_b) + bP_x(\tau_a > \tau_b)$

$f(x) = P_x(\tau_a < \tau_b)$ $af(x) + b(1 - f(x)) = x$

$f(x) = P_x(\tau_a < \tau_b) = \frac{b-x}{b-a}$

Law of the iterated logarithm (Khinchine)

B_t, $t \geq 0$ Brownian motion

$$\limsup_{t \to 0} \frac{B_t}{\sqrt{2t \log \log t^{-1}}} = 1 \qquad a.s.$$