Stochastic differential equations

$\sigma(x, t)$, $b(x, t)$ mble

Definition

A stochastic process X_t is a solution of a stochastic differential equation

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \qquad X_0 = x_0$$

on $[0, T]$ if X_t is progressively measurable with respect to $\mathcal{F}_t$, $\int_0^T |b(X_t, t)|dt < \infty$, $\int_0^T |\sigma(X_t, t)|^2 dt < \infty$ a.s. and

$$X_t = x_0 + \int_0^t b(X_s, s)ds + \int_0^T \sigma(X_s, s)dB_s \qquad 0 \leq t \leq T$$

The main point is that $\sigma(\omega, t) = \sigma(X_t, t)$, $b(\omega, t) = b(X_t, t)$

If $B(t)$ is a d-dimensional Brownian motion and $f(t, x)$ is a function on $[0, \infty) \times \mathbf{R}^d$ which has one continuous derivative in t and two continuous derivatives in x, then Ito's formula reads

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \nabla f((t, B(t)) \cdot dB(t) + \frac{1}{2}\Delta f(t, B(t))dt.$$

Bessel process ($d = 2$)

Let $B_t = (B_t^1, B_t^2)$ be 2d Brownian motion starting at 0,

$$r_t = |B_t| = \sqrt{(B_t^1)^2 + (B_t^2)^2}.$$

By Ito's lemma,

$$dr_t = \frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2 + \frac{1}{2}\frac{1}{|B|} dt.$$

This is *not* a stochastic differential equation.

$$Y(t) = \int_0^t \frac{B^1}{|B|} dB^1 + \int_0^t \frac{B^2}{|B|} dB^2$$

Let $f(t, y)$ be a smooth function   Use Itô's lemma. Intuitively

$$df(t, Y_t) = \partial_t f dt + \partial_y f dY + \frac{1}{2}\partial_y^2 f (dY)^2$$

$$\begin{aligned}
(dY)^2 &= \left(\frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2\right)^2 \\
&= \left(\frac{B^1}{|B|}\right)^2 (dB^1)^2 + 2\frac{B^1 B^2}{|B|^2} dB^1 dB^2 + \left(\frac{B^2}{|B|}\right)^2 (dB^2)^2 \\
&= dt
\end{aligned}$$

$$\begin{aligned}
f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s) ds \\
&\quad + \int_0^t \partial_y f \frac{B^1}{|B|} dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|} dB^2
\end{aligned}$$

Itô's lemma

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left\{ \partial_s f(s, X_s) + \mathcal{L} f(s, X_s) \right\} ds \\ &\quad + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

$$\mathcal{L} f(t, x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{d} b_i(t, x) \frac{\partial f}{\partial x_i}(t, x)$$

$$a_{ij} = \sum_{k=1}^{d} \sigma_{ik} \sigma_{jk} \qquad a = \sigma \sigma^T$$

$$dr_t = \frac{B^1}{|B|}dB^1 + \frac{B^2}{|B|}dB^2 + \frac{1}{2}\frac{1}{|B|}dt = dY_t + \frac{1}{2}r_t^{-1}dt$$

$$\begin{aligned} f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s)ds \\ &\quad + \int_0^t \partial_y f \frac{B^1}{|B|}dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|}dB^2 \end{aligned}$$

In particular $e^{\lambda Y_t - \lambda^2 t/2}$ is a martingale

So Y_t is a Brownian motion.

Therefore

$$dr_t = dY_t + \frac{1}{2}r_t^{-1}dt$$

is a stochastic differential equation for the new Brownian motion Y_t

Itô's lemma

$$f(t,X_t) - f(0,X_0) = \int_0^t \left\{\partial_s + \mathcal{L}\right\} f(s,X_s)ds + \int_0^t \nabla f(s,X_s)\cdot\sigma dB_s$$

Proof

$$\begin{aligned}
&= \sum_i f(t_{i+1}, X_{t_{i+1}}) - f(t_i, X_{t_i}) \\
&= \sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) + \nabla f(t_i, X_{t_i})\cdot(X_{t_{i+1}} - X_{t_i}) \\
&+ \frac{1}{2}\sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \\
&+ \text{ higher order terms}
\end{aligned}$$

Proof continued

$$\sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) \to \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds$$

$$\sum_i \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})$$

$$= \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} \sigma(s, X_s)dB_s) \quad \to \int_0^t \nabla f \cdot \sigma dB$$

$$+ \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} b(s, X_s)ds) \quad \to \int_0^t \nabla f \cdot b\, ds$$

$$\sum_i \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \to \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(s, X_s) a_{jk}(s, X_s)ds$$

Proof continued

To show the last convergence, ie

$$\sum_i g(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \to \int_0^t g(s, X_s)a_{jk}(s, X_s)ds$$

$$\begin{aligned} Z(t_i, t_{i+1}) &= (\int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}(s, X_s)dB^l_s)(\int_{t_i}^{t_{i+1}} \sum_m \sigma_{km}(s, X_s)dB^m_s) \\ &\quad - \int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}\sigma_{kl}(s, X_s)ds \end{aligned}$$

$$E[|Z(t_i, t_{i+1})|^2] = \mathcal{O}((t_{i+1} - t_i)^2)$$

Proof continued

$$\sum_i g(t_i, X_{t_i}) E[\int_{t_i}^{t_{i+1}} a_{jk}(s, X_s) ds] \to \int_0^t g(s, X_s) a_{ij}(s, X_s) ds$$

$$E[(\sum_i g(t_i, X_{t_i}) Z(t_i, t_{i+1}))^2] = \sum_{i,j} E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1})]$$

$$i < j \quad E[E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1}) \mid \mathcal{F}_{t_j}]] = 0$$

$$\begin{aligned} i = j \qquad & E[E[g^2(t_i, X_{t_i}) Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = E[g^2(t_i, X_{t_i}) E[Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = \mathcal{O}((t_{i+1} - t_i)^2) \end{aligned}$$

$$f(t,X_t) - f(0,X_0) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds = \int_0^t \nabla f(s,X_s)\cdot \sigma dB_s$$

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t,x)\frac{\partial}{\partial x_i} \quad = \textit{generator}$$

$M_t = f(t,X_t) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds$ is a martingale

$$\begin{aligned} 0 &= E[f(t,X_t) - f(s,X_s) - \int_s^t \Big\{\partial_u + \mathcal{L}\Big\} f(u,X_u)du \mid \mathcal{F}_s] \\ &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \Big\{\partial_u + \mathcal{L}\Big\} f(u,y)p(s,x,u,y)dydu, \qquad X_s = x \end{aligned}$$

For any f,

$$\begin{aligned} 0 &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \left\{ \partial_u + \mathcal{L} \right\} f(u,y)p(s,x,u,y)dydu \end{aligned}$$

Fokker-Planck (Forward) Equation

$$\begin{aligned} \frac{\partial}{\partial t} p(s,x,t,y) &= \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j} (a_{i,j}(t,y)p(s,x,t,y)) \\ &\quad - \sum_{i=1}^{d} \frac{\partial}{\partial y_i} (b_i(t,y)p(s,x,t,y)) \\ &= L_y^* p(s,x,t,y) \end{aligned}$$

$$\lim_{t \downarrow s} p(s,x,t,y) = \delta(y-x).$$

Kolmogorov (Backward) Equation

$$
\begin{aligned}
-\frac{\partial}{\partial s} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\frac{\partial^2 p(s,x,t,y)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{d} b_i(s,x)\frac{\partial p(s,x,t,y)}{\partial x_i} \\
&= L_x p(s,x,t,y)
\end{aligned}
$$

$$
\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x).
$$

Proof.

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \leq s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time t

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,z)dz$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$. We get in the limit that p satisfy the backward equations. $\square$

Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

Forward $$\frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y), \quad t > s,$$

$$p(s,x,s,y) = \delta(y - x)$$

Backward $$-\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y - x)$$

Existence and Uniqueness Theorem

$\sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be Borel measurable, $\exists A < \infty$,

$$\|\sigma(x,t)\| + |b(x,t)| \le A(1+|x|) \qquad x \in \mathbb{R}^d,\ 0 \le t \le T$$

and *Lipschitz*;

$$\|\sigma(x,t) - \sigma(y,t)\| + |b(x,t) - b(y,t)| \le A|x-y|.$$

$x_0 \in \mathbb{R}^d$ indep of B_t, $E[|x_0|^2] < \infty$.
Then there exists a unique solution X_t on $[0, T]$ to

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

and $E[\int_0^T |X_t|^2 dt] < \infty$.

Uniqueness means that if X_t^1 and X_t^2 are two solutions then

$$P(X_t^1 = X_t^2,\ 0 \le t \le T) = 1$$

Proof of Uniqueness

Suppose X_t^1 and X_t^2 are solutions

$$\begin{aligned} X_t^1 - X_t^2 &= \int_0^t (b(X_s^1, s) - b(X_s^2, s))ds + \int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s \\ &\quad + x_0^1 - x_0^2 \end{aligned}$$

$$E[|X_t^1 - X_t^2|^2] \leq 4E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2]$$

$$+4E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s|^2] + 4E[|x_0^1 - x_0^2|^2]$$

$$E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2] \leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds$$

$$\begin{aligned} E[\int_0^t |\sigma(X_s^1, s) - \sigma(X_s^2, s))|^2 ds] &= E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))ds|^2] \\ &\leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds \end{aligned}$$

Call $\phi(t) = E[|X_t^1 - X_t^2|^2]$

$$\phi(t) \le 8A^2 \int_0^t \phi(s)ds + 4\phi(0)$$

$$\Phi(t) = \int_0^t \phi(s)ds$$

$$(e^{-8A^2t}\Phi(t))' = (\Phi'(t) - 8A^2\Phi(t))e^{-8A^2t} \le 4\phi(0)e^{-8A^2t}$$

$$e^{-8A^2t}\Phi(t) \le 4\phi(0)$$

$$\phi(t) \le 8A^2\Phi(t) + 4\phi(0) \le 8e^{8A^2t}\phi(0)$$

$$E[|X_t^1 - X_t^2|^2] \le 8e^{8A^2t}E[|x_0^1 - x_0^2|^2]$$

For each $0 \le t \le T$, $X_t^1 = X_t^2$ a.s. so $X_t^1 = X_t^2$ for all rational $t \in [0, T]$ a.s. By continuity this implies that $X_t^1 = X_t^2$ for all $t \in [0, T]$ a.s.

Proof of Existence

$$X_0(t) \equiv x_0$$

$$X_n(t) = x_0 + \int_0^t \sigma(s, X_{n-1}(s))dB(s) + \int_0^t b(s, X_{n-1}(s))ds$$

$$
\begin{aligned}
& E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2] \\
\text{Doob's inequality} \qquad & \le 4E[\int_0^T \|\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s))\|^2 ds] \\
& \quad + TE[\int_0^T |b(s, X_{n-1}(s)) - b(s, X_{n-2}(s))|^2 ds] \\
& \le C \int_0^T E[|X_{n-1}(s) - X_{n-2}(s)|^2] ds \\
& \le CTE[\sup_{0\le t\le T} |X_{n-1}(t) - X_{n-2}(t)|^2]
\end{aligned}
$$

Proof.

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le CTE[\sup_{0\le t\le T}|X_{n-1}(t)-X_{n-2}(t)|^2]$$

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le \frac{(CT)^n}{n!}$$

$$P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}) \le 2^{2n}E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2]$$

summable

$$\text{Borel}-\text{Cantelli} \quad \Rightarrow \qquad P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}\ i.o.) = 0.$$

Hence for almost every ω, $X_n(t) = X_0(t) + \sum_{j=0}^{n-1}(X_{j+1}(t) - X_j(t))$ converges uniformly on $[0, T]$ to a limit $X(t)$ which solves the required stochastic integral equation □

Lipschitz condition is *not* necessary

Theorem

Let $d = 1$ and

$$
\begin{aligned}
|b(t,x) - b(t,y)| &\leq C|x-y| \\
|\sigma(t,x) - \sigma(t,y)| &\leq C|x-y|^{\alpha}, \quad \alpha \geq 1/2
\end{aligned}
$$

Then there exists a solution of $dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$ and it is unique

But you do need *some* regularity

$\sigma(x) = \mathrm{sgn}(x)$ and $dX = \sigma(B)dB$ Not a stochastic differential equation
But X is a Brownian motion $dB = \sigma(B)dX$ *is* a stochastic differential equation
But also $d(-B) = \sigma(-B)dX$ so no uniqueness