# Conditional Expectation

Probability space $(\Omega, \mathcal{F}, P)$
Random variable $X \in L^1$
Sub $\sigma-$field $\mathcal{G} \subset \mathcal{F}$

## Definition

The conditional expectation of $X$ given $\mathcal{G}$ is a random variable $E[X|\mathcal{G}]$ satisfying

1. $E[X|\mathcal{G}] \in \mathcal{G}$
2. $\int_A X dP = \int_A E[X|\mathcal{G}] dP$ for all $A \in \mathcal{G}$

$X \geq 0$, $Q(A) = \int_A X dP$, $A \in \mathcal{G}$. $Q$ measure on $(\Omega, \mathcal{G}, P)$, $Q << P$

Radon-Nikodym theorem: $\exists \frac{dQ}{dP} \in L^1(\Omega, \mathcal{G}, P)$ s.t. $Q(A) = \int_A \frac{dQ}{dP} dP$

$E[X|\mathcal{G}] = \frac{dQ}{dP}$

if $X = X_+ - X_-$ define $E[X|\mathcal{G}] = E[X_+|\mathcal{G}] - E[X_-|\mathcal{G}]$

## Examples.

1. $\mathcal{G} = \{\emptyset, \Omega\}$, $E[X|\mathcal{G}] = E[X]$
2. $\mathcal{G} = \mathcal{F}$, $E[X|\mathcal{G}] = X$
3. $\Omega = [0,1)$, $\mathcal{F} =$Borel sets, $P =$ Lebesgue, $\mathcal{F}_n =$ Dyadic level n, $E[X|\mathcal{F}_n](\omega) = Av_{[\frac{i}{2^n}, \frac{i+1}{2^n})}X$, $\omega \in [\frac{i}{2^n}, \frac{i+1}{2^n})$
4. $A_1, A_2, \ldots$ partition of $\Omega$. $\mathcal{G}$ is $\sigma-$field generated by this partition $E[X|\mathcal{G}] = \frac{1}{P(A_i)} \int_{A_i} XdP$, $\omega \in A_i$

   In particular if $A_1 = A$, $A_2 = A^C$, $X = 1_B$ then $E[1_B|\mathcal{G}] = P(B|A) = \frac{P(B \cap A)}{P(A)}$ on $A$
5. $P((X,Y) \in A) = \int_A f(x,y)dxdy$ $E[g(X)|Y] = \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx} = \int g(x)P(X \in dx|Y = y)$
6. $X \in G \Rightarrow E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$
7. $X$ indep of $\mathcal{G} \Leftrightarrow E[X|\mathcal{G}] = E[X]$

## Examples.

8. $\phi$ convex then $E[\phi(X)|\mathcal{G}] \geq \phi(E[X|\mathcal{G}])$
9. $E[E[X|\mathcal{G}_1]|\mathcal{G}_2] = E[X|\mathcal{G}_2]$ if $\mathcal{G}_2 \subset \mathcal{G}_1$

# Markov processes

A process $X_t,\ t \geq 0$ is called a Markov process if for any function $g$ and any $t \geq s$,

$$E[g(X_t) \mid X_u,\ 0 \leq u \leq s] = E[g(X_t) \mid X_s].$$

Process determined by initial distr $P(X_0 \in A)$ and the transition probs

$$p(s, x, t, A) = P(X_t \in A \mid X_s = x) \qquad s < t$$

$$\begin{aligned} &P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n) \\ &= \int_{A_{n-1}} \cdots \int_{A_1} \int P(X_0 \in dx_0) p(0, x_0, t_1, dx_1) \cdots p(t_{n-1}, x_{n-1}, t_n, A_n) \end{aligned}$$

Chapman-Kolmogorov equations

$$p(s, x, t, A) = \int p(s, x, u, dy) p(u, y, t, A) \qquad \text{for } s \leq u \leq t.$$

Example. Brownian motion $p(s, x, t, dy) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} dy$

# Brownian motion as limit of random walks

$S_n = X_1 + \cdots + X_n$, $X_i$ iid, $E[X_i] = 0$, $Var(X_i) = \sigma^2 < \infty$

$$B_n(t) = n^{-1/2} S_{\lfloor nt \rfloor}$$

$F^n_{t_1,\dots,t_m}(x_1,\dots,x_m) = P(B_n(t_1) \le x_1, \dots, B_n(t_m) \le x_m)$

$$\lim_{n\to\infty} F^n_{t_1,\dots,t_m}(x_1,\dots,x_m) =$$

$$\int_{-\infty}^{x_m} \cdots \int_{-\infty}^{x_1} \frac{e^{-\frac{y_1^2}{2t_1}}}{\sqrt{2\pi t_1}} \frac{e^{-\frac{(y_2-y_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\frac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}}}{\sqrt{2\pi(t_m-t_{m-1})}} dy_1 \cdots dy_m$$

$\{B_n(t_i) - B_n(t_{i-1})\}_{i=1,\dots,m} = \{n^{-1/2}(X_{\lfloor nt_{i-1} \rfloor+1} + \cdots + X_{\lfloor nt_i \rfloor})\}_{i=1,\dots,m}$

multidimensional central limit theorem

# Martingales: Discrete time

**Definition.**

An non-decreasing family of sub-$\sigma$-fields $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ is called a *filtration*

**Definition**

$M_n$ a sequence of random variables in $L^1(\Omega, \mathcal{F}, P)$. If

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n$$

then $M_n$ is a *martingale* with respect to the filtration $\mathcal{F}_n$

submartingale: $E[M_{n+1} \mid \mathcal{F}_n] \geq M_n$

supermartingale: $E[M_{n+1} \mid \mathcal{F}_n] \leq M_n$

**Example.** $S_n = X_1 + \cdots + X_n$, $X_i$ iid

$E[X_i] = 0 \Rightarrow S_n$ martingale. $E[X_i] \geq 0 \Rightarrow S_n$ submartingale.
$E[X_i] \leq 0 \Rightarrow S_n$ supermartingale

## Lemma

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and $X_n$ a martingale with respect to $\mathcal{F}_n$. Then $\phi(X_n)$ is a submartingale with respect to $\mathcal{F}_n$.

## Proof.

By Jensen's inequality for conditional probability

$$E[\phi(X_{n+1}) \mid \mathcal{F}_n] \geq \phi(E[X_{n+1} \mid \mathcal{F}_n]) = \phi(X_n).$$

□

## Example. $S_n = X_1 + \cdots + X_n$, $X_i$ iid, $E[X_i] = 0$, $Var(X_i) = \sigma^2 < \infty$

$S_n$ martingale. $S_n^2$ submartingale. $S_n^2 - \sigma^2 n$ martingale.

$E[S_{n+1}^2 - \sigma^2(n+1)|\mathcal{F}_n] = E[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - \sigma^2(n+1)|\mathcal{F}_n] = S_n^2 + \sigma^2 - \sigma^2(n+1)$