Martingale representation theorem

$\Omega = C[0, T]$, $\mathcal{F}_T$ = smallest σ-field with respect to which B_s are all measurable, $s \leq T$, P the Wiener measure , B_t = Brownian motion

M_t square integrable martingale with respect to $\mathcal{F}_t$

Then there exists $\sigma(t, \omega)$ which is

1. progressively measurable
2. square integrable
3. $\mathcal{B}([0, \infty)) \times \mathcal{F}$ mble

such that

$$M_t = M_0 + \int_0^t \sigma(s) dB_s$$

Lemma

$\mathcal{A}$ = set of all linear combinations of random variables of the form

$$e^{\int_0^T h dB - \frac{1}{2}\int_0^T h^2 dt}, \qquad h \in L^2([0,T])$$

$\mathcal{A}$ is dense in $L^2(\Omega, \mathcal{F}_T, P)$

Proof

Suppose $g \in L^2(\Omega, \mathcal{F}_T, P)$ is orthogonal to all such functions

We want to show that $g = 0$

By an easy choice of simple functions h we find that for any $\lambda_1, \ldots, \lambda_n \in R$ and $t_1, \ldots, t_n \in [0,T]$,

$$E^P[g e^{\lambda_1 B_{t_1} + \cdots + \lambda_n B_{t_n}}] = 0$$

lhs real analytic in λ and hence has an analytic extension to $\lambda \in \mathbb{C}^n$

Since $E^P[ge^{\lambda_1 B_{t_1}+\cdots+\lambda_n B_{t_n}}]$ is analytic and vanishes on the real axis, it is zero everywhere. In particular

$$E^P[ge^{i(y_1 B_{t_1}+\cdots+y_n B_{t_n})}] = 0$$

Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\hat{\phi}(y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \phi(x) e^{-ix\cdot y} dx$$

Fourier inversion:

$$\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\phi}(y) e^{ix\cdot y} dy$$

$$E^P[g\phi(B_{t_1},\ldots,B_{t_n})] = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\phi}(y) E^P[e^{iy_1 B_{t_1}+\cdots+y_n B_{t_n})}] dy = 0$$

Hence g is orthogonal to fns of form $\phi(B_{t_1},\ldots,B_{t_n})$ where $\phi \in C_0^\infty(\mathbb{R}^n)$
Dense in $L^2(\Omega,\mathcal{F}_T,P) \quad \Rightarrow \quad g = 0$

Lemma

$F \in L^2(\Omega, \mathcal{F}_T, P)$ There exists a unique $f(t,\omega)$ which is

1. progressively measurable
2. square integrable
3. $\mathcal{B}([0,\infty)) \times \mathcal{F}$ measurable

such that

$$F(\omega) = E[F] + \int_0^T f dB.$$

Proof of Uniqueness

suppose

$$F = E[F] + \int_0^T f_1 dB = E[F] + \int_0^T f_2 dB$$

$$\Rightarrow \int_0^T (f_2 - f_1) dB = 0 \quad \Rightarrow \int_0^T E[(f_2 - f_1)^2] dt = 0 \quad \Rightarrow \quad f_2 = f_1$$

Proof of existence

First we prove it if F is of the form $F = e^{\int_0^T h dB - \frac{1}{2}\int_0^T h^2 ds}$
Defining $F_t = e^{\int_0^t h dB - \frac{1}{2}\int_0^t h^2 ds}$ gives

$$dF = hFdB, \qquad F_0 = 1,$$

so

$$F_t = 1 + \int_0^t F_s h dB.$$

Plugging in $t = T$ gives the result.

If F is a linear combination of such functions the result follows by linearity

Proof of existence for $F \in L^2(\Omega, \mathcal{F}_T, P)$

$F_n \in L^2(\Omega, \mathcal{F}_T, P)$ with $F_n \to F$ and

$$F_n = E[F_n] + \int_0^T f_n dB.$$

$E[F_n] \to E[F]$, so wlog $E[F_n] = E[F] = 0$

$$E[(F_n - F_m)^2] = \int_0^T E[(f_n - f_m)^2]dt \to 0 \quad as \quad n, m \to \infty$$

$\Rightarrow \quad f_n$ Cauchy in $L^2([0, T] \times \Omega, dx \times dP)$.

Let f be the limit. Taking limits we have

$$F = E[F] + \int_0^T f dB.$$

Proof of the martingale representation theorem

By previous lemma, for each t we have $\sigma_t(s,\omega)$ such that

$$M_t = E[M_t] + \int_0^t \sigma_t(s)dB_s$$

Let $t_2 > t_1$

$$M_{t_1} = E[M_{t_2} \mid \mathcal{F}_{t_1}]$$

$$\int_0^{t_1} \sigma_{t_2}(s)dB_s = \int_0^{t_1} \sigma_{t_1}(s)dB_s$$

Uniqueness $\Rightarrow \quad \sigma_{t_1} = \sigma_{t_2}$

Quadratic variation of $X_t = \int_0^t \sigma(s)dB_s$

$e^{\lambda \int_0^t \sigma(s)dB_s - \frac{\lambda^2}{2}\int_0^t \sigma^2(s)ds}$ =martingale

$$E[e^{\lambda \int_{t_i}^{t_{i+1}} \sigma(s)dB_s - \frac{\lambda^2}{2}\int_{t_i}^{t_{i+1}} \sigma^2(s)ds} \mid \mathcal{F}_{t_i}] = 0$$

$$E[Z(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}] = 0, \qquad Z(t_i, t_{i+1}) = (\int_{t_i}^{t_{i+1}} \sigma(s)dB_s)^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s)ds$$

$$E[\{Z(t_i, t_{i+1})\}^2 - 4(\int_{t_i}^{t_{i+1}} \sigma^2(s)ds)^2 \mid \mathcal{F}_{t_i}] = 0$$

$$E[(\sum_i Z(t_i, t_{i+1}))^2] \leq 4E[(\sum_i (\int_{t_i}^{t_{i+1}} \sigma^2(s)ds)^2] \to 0$$

$$\langle X_t, X_t \rangle = \int_0^t \sigma^2(s, \omega)ds$$

Levy's Theorem

Let X_t be a process adapted to a filtration $\mathcal{F}_t$ which

1. has continuous sample paths
2. is a martingale
3. has quadratic variation t

Then X_t is a Brownian motion

Proof of Levy's theorem

Enough to show that for each λ,

$$E[e^{i\lambda(X_t - X_s)} \mid \mathcal{F}_s] = e^{-\frac{1}{2}\lambda^2(t-s)}$$

Call $M_t = e^{i\lambda X_t + \frac{1}{2}\lambda^2 t}$, $t_j = s + \frac{j}{2^n}(t-s)$

$$M_t - M_s = \sum_{j=1}^{2^n} M_{t_j} - M_{t_{j-1}}$$

$$= \sum_{j=1}^{2^n} i\lambda M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) - \frac{1}{2}\lambda^2 M_{\xi_j}[(X_{t_j} - X_{t_{j-1}})^2 - (t_j - t_{j-1})]$$

$$E[M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_s] = E[E[M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}}] \mid \mathcal{F}_s]$$

$$= E[M_{t_{j-1}} E[(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}}] \mid \mathcal{F}_s] = 0$$

Proof of Levy's theorem

Fix m. Let $\xi^m = \max\{\frac{i}{2^m} : \frac{i}{2^m} \leq \xi\}$

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} M_{\xi_j^m}[(X_{t_j} - X_{t_{j-1}})^2 - (t_j - t_{j-1})] = 0$$

So we only have to show

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} [M_{\xi_j^m} - M_{\xi_j}](X_{t_j} - X_{t_{j-1}})^2 = 0$$

Would follow from

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} (X_{\xi_j^m} - X_{\xi_j})(X_{t_j} - X_{t_{j-1}})^2 = 0$$

Left hand side $= t \lim_{n\to\infty} \max_{1\leq j\leq 2^n} |X_{\xi_j^m} - X_{\xi_j}| = 0$ a.s.

Note the same proof gives

Itô formula for semimartingales

Let $M_t^1, \ldots, M_t^d$ be martingales with respect to a filtration $\mathcal{F}_t$, $t \geq 0$, $A_t^1, \ldots, A_t^d$ adapted processes of bounded variation, $X_t = x_0 + A_t + M_t$ where $x_0 \in \mathcal{F}_0$, and $f(t,x) \in C^{1,2}$. Then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dA_s^i + dM_s^i \\ &\quad + \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d\langle M^i, M^j \rangle_s \end{aligned}$$

Lemma

Suppose P probability measure on Ω, $\mathcal{F}_t$ such that

$$M_\lambda(t) = \exp\{\lambda(X(t) - X(0) - \int b(s, X(s))ds) - \frac{\lambda^2}{2} a(s, X(s))ds\}$$

are martingales for every λ. Then there is a Brownian motion $B(t)$ such that

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$$

- Only stated in $d = 1$ but analogous result in $d > 1$
- In the proof we will assume a, b smooth, bounded and $a \geq \epsilon > 0$ but it is not necessary

1. Write $\tilde{X}(t) = X(t) - \int_0^t b(s, X(s))ds$
2. Differentiate in λ and evaluate at $\lambda = 0$ to see that $\tilde{X}(t)$ is a martingale
3. To justify this, note that $E[e^{\lambda \tilde{X}(t)}] \leq e^{C\lambda^2 t}$ so by Tchebyshev $P(\tilde{X}(t) \geq x) \leq e^{-C' x^2}$
4. Differentiate in λ again to see that $\tilde{X}^2(t) - \int_0^t a(s, X(s))ds$ is a martingale
5. Let $\sigma = \sqrt{a}$ and define $B(t) = \int_0^t \frac{1}{\sigma(s, X(s))} d\tilde{X}(s)$
6. Then $B(t)$ is a martingale with quadratic variation t
7. Levy's theorem $\Rightarrow$ $B(t)$ is a Brownian motion
8. $dX(t) = d(\tilde{X}(t) + \int_0^t b(s, X(s))ds) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$

Passive tracer in Burgers flow

Burgers equation

$$\partial_t u = u\partial_x u + \frac{1}{2}\partial_x^2 u + \partial_x F, \qquad u(0,x) = u_0(x),\ F = F(t,x)$$

Passive tracer

$$dX = udt + dB(t)$$

Solution for u by Hopf-Cole transformation: $\partial_x h = u$

$$\partial_t h = \frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + F$$

$$Z(t,x) = e^{h(t,x)}$$

$$dZ = \partial_x^2 Z + FZ$$

$$\partial_t u = u\partial_x u + \frac{1}{2}\partial_x^2 u + \partial_x F, \qquad u(0,x) = u_0(x),\ F = F(t,x)$$

$$\partial_x h = u \qquad Z(t,x) = e^{h(t,x)}$$

$$dZ = \partial_x^2 Z + FZ$$

$$h(t,x) = \log E_x[\exp\{\int_0^t F(t-s, B(s))ds + h(0, B(T))\}]$$

Passive tracer

$$dX = u(T-t, X(t))dt + dB(t)$$

Cameron-Martin-Girsanov derivative wrt $B(t)$ is

$$\exp\{\int_0^T u dB(s) - \frac{1}{2}\int_0^T u^2 ds\}$$

Ito's formula applied to $h(T-t, B(t))$

$$\begin{aligned} dh &= \{-\partial_t + \frac{1}{2}\partial_x^2\}h dt + \partial_x h dB \\ &= (-\frac{1}{2}u^2 - F)dt + u dB \end{aligned}$$

So Cameron-Martin-Girsanov becomes

$$\exp\{-h(T, B(0)) + h(0, B(T)) + \int_0^T F(T-s, B(s))ds\}$$

In particular,

$$E_x[f(X(T))] = E_x[e^{-h(T,B(0))+h(0,B(T))+\int_0^T F(T-s,B(s))ds} f(B(T))]$$

$$E_x[f(X(T))] = E_x[e^{-h(T,x)+h(0,B(T))+\int_0^T F(T-s,B(s))ds} f(B(T))]$$

write in terms of Brownian Bridge

$$= \int E_{x,y}[e^{-h(T,x)+h(0,y)+\int_0^T F(T-s,B(s))ds} f(y)] P_x(B(T) \in dy)$$

$$E_{xy}[g(B(\cdot))] = E_x[g(B(\cdot)) \mid B(T) = y]$$

Easy to see it is a Markov process because if $t > s$

$$E_{xy}[h(B(t)) \mid \mathcal{F}(s)] = E_x[h(B(t)) \mid \mathcal{F}_s, B(T) = y] = E_x[h(B(t)) \mid B(s), B(T)$$

$$= \int h(x) p(t-s, x-B(s)) dx p(T-t, y-x)$$