Stochastic Calculus

Lecture 2

Theorem (Kolmogorov Regularity Theorem)

Let $X(t,\omega)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that for some positive constants α, β, C,

$$E[|X(t) - X(s)|^{\beta}] \leq C|t - s|^{1+\alpha}.$$

Then there is a version $\tilde{X}(t)$ of $X(t)$ on $C[0, T]$, by which we mean that $\tilde{X}$ and X have the same finite dimensional distributions.

Example. Brownian motion

$$E[|B(t) - B(s)|^4] = 3|t - s|^2$$

Counterexample. Poisson process

$$E[|N(t) - N(s)|^{\alpha}] = \sum_{n=0}^{\infty} n^{\alpha} \frac{|t - s|^n}{n!} e^{-|t-s|} \geq 1 - e^{-|t-s|} \simeq |t - s|$$

Theorem (Kolmogorov Regularity Theorem)

Let $X(t,\omega)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that for some positive constants α, β, C,

$$E[|X(t) - X(s)|^{\beta}] \le C|t - s|^{1+\alpha}.$$

Then there is a version $\tilde{X}(t)$ of $X(t)$ on $C[0, T]$, by which we mean that $\tilde{X}$ and X have the same finite dimensional distributions.
Furthermore X is Hölder of order $\gamma < \alpha/\beta$, i.e.

$$|X(t) - X(s)| \le C|t - s|^{\gamma} \qquad 0 \le s, t \le T$$

Example. Brownian motion

$$E[|B(t) - B(s)|^{2m}] = C_m|t - s|^{m}$$

so $B(t)$ is Hölder of order γ for any $\gamma < 1/2$.

Key point of the proof of Kolmogorov's theorem

$$A_n = \{|X(\frac{i+1}{2^n}) - X(\frac{i+1}{2^n})| \geq 2^{-\gamma n} \text{ for some} i = 0, \dots, 2^n - 1\}$$

$$P(A_n) \leq \sum_{i=0}^{2^n-1} P(|X(\frac{i+1}{2^n}) - X(\frac{i+1}{2^n})| \geq 2^{-\gamma n})$$

$$P(|X(\frac{i+1}{2^n}) - X(\frac{i+1}{2^n})| \geq 2^{-\gamma n}) \leq 2^{\beta\gamma n} E[|X(\frac{i+1}{2^n}) - X(\frac{i+1}{2^n})|^\beta]$$

$$P(A_n) \leq C 2^{n(-\alpha+\gamma\beta)}$$

summable if $\gamma < \alpha/\beta$ so by Borel-Cantelli A_n only happens finitely many times.

Functions of finite variation

$f : \mathbb{R}_+ \to \mathbb{R}$ right continuous

$\Delta = 0 = t_0 < t_1 < \cdots < t_n = t$ subdivision of $[0, t]$

$|\Delta| = \sup_i |t_{i+1} - t_i| =$ mesh size

f is of finite variation if for each $t < \infty$,
$\|f\|_{TV,[0,t]} = \sup_\Delta \sum_i |f(t_{i+1}) - f(t_i)| < \infty$

Proposition

1. *A function of finite variation is the difference of two monotone increasing functions*
2. $\mu([0, t]) = f(t)$ *provides a 1-1 correspondence between measures on* $\mathbb{R}_+$ *and functions of finite variation*
3. $\int_0^\infty g(t)df(t) = \int g d\mu$ *is the Riemann-Stieltjes integral*
4. *A function of finite variation is differentiable almost everywhere*

Quadratic variation

Definition

A stochastic process X_t, $t \geq 0$ has finite quadratic variation
if there exists a finite process $\langle X, X \rangle_t$, $t \geq 0$ such that for each $t < \infty$
and each sequence $\{\Delta_n\}_{n=1,2,\ldots}$ of subdivisions of $[0, t]$ with $|\Delta_n| \to 0$,

$$\lim_{n\to\infty} \sum_i |X_{t_{i+1}} - X_{t_i}|^2 \overset{prob}{=} \langle X, X \rangle_t$$

The process $\langle X, X \rangle_t$, $t \geq 0$ is non-decreasing.
It is called the *quadratic variation* of X.

Recall $\lim_{n\to\infty} X_n \overset{prob}{=} X$ if $P(|X_n - X| \geq \epsilon) \to 0$ for each $\epsilon > 0$

Theorem

The quadratic variation of Brownian motion $\langle B, B\rangle_t = t$

Proof.

$P(X_n \geq \epsilon) \leq \frac{E[X_n^2]}{\epsilon^2}$ so it is enough to show

$$E[(\sum_i |B_{t_{i+1}} - B_{t_i}|^2 - |t_{i+1} - t_i|)^2] \to 0$$

Call $X_i = |B_{t_{i+1}} - B_{t_i}|^2 - |t_{i+1} - t_i|$. They are independent so

$$E[(\sum_i X_i)^2] = \sum_i E[X_i^2]$$

$X_i = (t_{i+1} - t_i)(Z_i^2 - 1)$ where $Z_i \sim \mathcal{N}(0, 1)$ so

$$E[X_i^2] = (t_{i+1} - t_i)^2 E[(Z_i^2 - 1)^2] = C(t_{i+1} - t_i)^2$$

so $E[(\sum_i X_i)^2] = C \sum_i (t_{i+1} - t_i)^2 \to 0$ $\square$

Quadratic variation

Note that if $X_n \overset{prob}{\rightarrow} X$ one can always choose a (non-random) subsequence such that $X_n \overset{a.s.}{\rightarrow} X$
So one can choose partitions so that

$$\lim_{n\to\infty} \sum_i |X_{t_{i+1}} - X_{t_i}|^2 \overset{a.s.}{=} \langle X, X \rangle_t$$

For Brownian motion it turns out that any sequence $\Delta_n \subset \Delta_{n+1}$ gives a.s. convergence

Proposition

With probability one Brownian motion B_t, $t \geq 0$ is not of finite variation in any interval

Proof.

Let f be any continuous function on $[0, t]$

$$\sum_i |f_{t_{i+1}} - f_{t_i}|^2 \leq \max_i |f_{t_{i+1}} - f_{t_i}| \sum_i |f_{t_{i+1}} - f_{t_i}|$$

Since $\max_i |f_{t_{i+1}} - f_{t_i}| \to 0$, if

$$\langle f, f \rangle_t = \lim \sum_i |f_{t_{i+1}} - f_{t_i}|^2 > 0$$

then

$$\lim \sum_i |f_{t_{i+1}} - f_{t_i}| = \|f\|_{TV,[0,t]} = \infty$$

and if $\|f\|_{TV,[0,t]} < \infty$ then $\langle f, f \rangle_t = 0$ ☐

Proposition

With probability one Brownian motion B_t, $t \geq 0$ is not locally Hölder of order α for any $\alpha > 1/2$

Proof.

Let f be any continuous function on $[0, t]$ s.t. for some $0 \leq a < b \leq t$ and some $\alpha > 1/2$, for all $a \leq s, t \leq b$,

$$|f_t - f_s| \leq k|t - s|^{\alpha}$$

Then

$$\sum_i |f_{t_{i+1}} - f_{t_i}|^2 \leq k^2(b - a) \max_i |t_{i+1} - t_i|^{2\alpha - 1}$$

□

Theorem. (Paley, Wiener, Zygmund 33)

Brownian motion is nowhere differentiable with probability one

Proof. (Dvoretsky, Erdös, Kakutani 61)

Suppose that $B(t)$ was differentiable at a point $s \in [0, 1]$.

Then $\exists \epsilon > 0$ and an integer $\ell \geq 1$ such that

$$|B(t) - B(s)| \leq \ell(t - s) \qquad \text{for} \quad 0 < t - s < \epsilon.$$

Choose an integer $n > \ell$ large enough so that

$$s \leq \frac{i}{n} < \frac{i+1}{n} < \frac{i+2}{n} < \frac{i+3}{n} < s + \epsilon \qquad \text{where} \quad i = \lfloor ns \rfloor + 1.$$

Then

$$|B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \qquad \text{for} \quad j = i+1, i+2, i+3.$$

Proof.

Therefore the event that $B(t)$ is differentiable at some point is contained in the set

$$B = \bigcup_{\ell \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} \bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\}.$$

We show $P(B) = 0$ as follows.

$$P\left(\bigcap_{n \geq m} \bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\} \right)$$

$$\leq \liminf_{n \to \infty} P\left(\bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\} \right)$$

Proof.

$$
\begin{aligned}
&P\left(\bigcap_{n\geq m}\bigcup_{0\leq i\leq n+1}\bigcap_{i\leq j\leq i+3}\left\{|B(\frac{j}{n})-B(\frac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty} P\left(\bigcup_{0\leq i\leq n+1}\bigcap_{i\leq j\leq i+3}\left\{|B(\frac{j}{n})-B(\frac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty}\sum_{i=1}^{n+1} P\left(\bigcap_{i\leq j\leq i+3}\left\{|B(\frac{j}{n})-B(\frac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty} n\left[P\left(|B(\frac{1}{n})|<\frac{7\ell}{n}\right)\right]^3\\
&= \liminf_{n\to\infty} n\left[P\left(|B(1)|<\frac{7\ell}{\sqrt{n}}\right)\right]^3 = \liminf_{n\to\infty} n\left[\frac{7\ell}{\sqrt{n}}\right]^3 = 0
\end{aligned}
$$

☐

Modulus of continuity (P.Levy)

With probability one,

$$\limsup_{\epsilon \to 0} \sup_{\substack{0 \leq s < t \leq 1 \\ t-s<\epsilon}} \frac{|B_t - B_s|}{\sqrt{2\epsilon \log \epsilon^{-1}}} = 1$$

Lemma $\frac{x}{x^2+1} e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{y^2}{2}} dy \leq x^{-1} e^{-\frac{x^2}{2}} \qquad x > 0$

Proof.

$$x^{-2} \int_x^\infty e^{-\frac{y^2}{2}} dy \geq \int_x^\infty y^{-2} e^{-\frac{y^2}{2}} dy = x^{-1} e^{-\frac{x^2}{2}} - \int_x^\infty e^{-\frac{y^2}{2}} dy$$

$$\int_x^\infty e^{-\frac{y^2}{2}} dy \leq x^{-1} \int_x^\infty y e^{-\frac{y^2}{2}} dy = x^{-1} e^{-\frac{x^2}{2}}$$

□

Proof of $\limsup_{\epsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s<\epsilon}} \frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}} \ge 1$

let $\delta > 0$, $A_n = \{\max_{1\le k\le 2^n} |B_{\frac{k}{2^n}} - B_{\frac{k-1}{2^n}}| \le (1-\delta)h(2^{-n})\}$,
$h(t) = \sqrt{2t\log t^{-1}}$

$$P(A_n) \le (1 - 2\int_{(1-\delta)\sqrt{2\log 2^n}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy)^{2^n} \quad \textit{independent increments}$$

$$\le e^{-Cn^{-1/2}2^{n(1-(1-\delta)^2)}} \qquad \textit{by lemma}$$

so $\sum_{n=1}^{\infty} P(A_n) < \infty$. By the Borel-Cantelli lemma, almost every ω is in at most finitely many A_n.i.e.

$$\limsup_{\epsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s<\epsilon}} \frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}} \ge 1-\delta$$

now let $\delta \downarrow 0$

Note: This proves that Brownian motion is not Hölder of order $\alpha \ge 1/2$

Proof of $\lim\sup_{\epsilon\to 0}\sup_{\substack{0\le s<t\le 1\\ t-s<\epsilon}}\frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}}\le 1$

let $\delta > 0$ and choose $\epsilon > 0$ so that $(1+\epsilon)^2(1-\delta) > 1+\delta$
let

$$B_n = \{\max_{i,j\in K}\frac{|B_{j/2^n} - B_{i/2^n}|}{h(k/2^n)} \ge 1+\epsilon\}$$

$$K = \{0 \le i < j < 2^n, 0 < k = j - i \le 2^{n\delta}\}$$

$$\begin{aligned} P(B_n) &\le \sum_K \frac{2}{\sqrt{2\pi}}\int_{(1+\epsilon)\sqrt{\log(k^{-1}2^n)}}^{\infty} e^{-\frac{y^2}{2}}dy \\ &\le C\sum_K [\log(k^{-1}2^n)]^{-1/2} e^{-(1+\epsilon)^2\log(k^{-1}2^n)} \quad \textit{lemma} \\ &\le C2^{-n(1-\delta)(1-\epsilon)^2}\sum_K [\log(k^{-1}2^n)]^{-1/2} \quad k^{-1}\ge 2^{-n\delta} \end{aligned}$$

$$B_n = \{\max_{i,j\in K} \frac{|B_{j/2^n} - B_{i/2^n}|}{h(k/2^n)} \geq 1 + \epsilon\}$$

$$K = \{0 \leq i < j < 2^n, 0 < k = j - i \leq 2^{n\delta}\}$$

$$P(B_n) \leq C2^{-n(1-\delta)(1-\epsilon)^2} \sum_K [\log(k^{-1}2^n)]^{-1/2}$$

$$\leq Cn^{-1/2}2^{n((1+\delta)-(1-\delta)(1+\epsilon)^2)} \quad |K| \leq 2^{n(1+\delta)}, \log(k^{-1}2^n) \geq \log 2^{n(1-\delta)}$$

summable so with probability one there is an N s.t. for $n > N$, for

$i, j \in K$,

$$|B_{j/2^n} - B_{i/2^n}| < (1 + \epsilon)h(k/2^n)$$

Let $\gamma > 0$. Pick N large so that $\sum_{m=n+1}^{\infty} h(2^{-m}) \leq \gamma h(2^{-(n+1)(1-\delta)})$, $n \geq N$

Suppose that $t = i2^{-n} + 2^{-n_1} + 2^{-n_2} + \cdots$ with $N \leq n < n_1 < n_2 < \cdots$ and $i \in K$,

$$|B_t - B_{i/2^n}| \leq (1+\epsilon) \sum_{m=n+1}^{\infty} h(2^{-m}) \leq (1+\epsilon)\gamma h(2^{-(n+1)(1-\delta)})$$

now suppose we have $0 \leq s < t \leq 1$ and the special n so that

$$2^{-(n+1)(1-\delta)} \leq t - s < 2^{-n(1-\delta)}$$

has $n \geq N$ then we can write

$$\begin{aligned} |B_t - B_s| &\leq |B_{i/2^n} - B_s| + |B_{j/2^n} - B_{i/2^n}| + |B_t - B_{j/2^n}| \\ &\leq 2(1+\epsilon)\gamma h(2^{-(n+1)(1-\delta)}) + (1+\epsilon)h((j-i)2^{-n}) \\ &\leq (2(1+\epsilon)\gamma + 1 + \epsilon)h(t-s) \qquad \textit{if } t-s \textit{ small enough} \end{aligned}$$

let $\delta \downarrow 0$ and then $\gamma \downarrow 0$