Brownian motion in $\mathbb{R}^d$

1. $B_t = (B_t^1, \ldots, B_t^d)$, B_t^i independent Brownian motions
2. B_t Markov with $P(B_t \in A \mid B_s = x) = \int_A \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|y-x|^2}{2(t-s)}} dy$
3. B_t has stationary independent mean zero increments with $E[|B_t - B_s|^2] = d(t-s)$
4. $e^{\lambda \cdot B_t - \frac{1}{2}|\lambda|^2 t}$ is a martingale for any λ

Note that 1 does not depend on the basis: If $B_t^1, \ldots, B_t^2$ independent and $\mathcal{O}$ is orthogonal, then the coordinates of $\mathcal{O}B_t$ are independent Brownian motions in fact

Theorem

Suppose X_1, X_2 independent and $\exists \theta \neq N\pi/2$ such that

$$X_1 \cos\theta + X_2 \sin\theta, \qquad -X_1 \sin\theta + X_2 \cos\theta \qquad \text{independent}$$

Then X_1, X_2 are Gaussians (Maxwell)

Dirichlet problem

Given a bounded open subset $G \subset \mathbf{R}^d$ and a continuous function $f : \partial G \to \mathbf{R}$ find a continuous function $u : \bar{G} \to R$ such that

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u|_{\partial G} = f \end{cases}$$

$$\Delta u \stackrel{def}{=} \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 2d \lim_{r \to 0} r^{-2} \left(\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS - u(x) \right)$$

Lemma

u harmonic in $G \Leftrightarrow u$ satisfies the mean value property: for all sufficiently small $r > 0$,

$$\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS = u(x)$$

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Proof.

Green's identity $\int_G v \Delta u dx = \int_G u \Delta v dx + \int_{\partial G} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \ dS$

$G = \{\delta < x < r\}$, $\quad v = \begin{cases} \frac{\log r - \log |x|}{\log r - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - r^{2-d}}{\delta^{2-d} - r^{2-d}} & d > 2 \end{cases}$

let $\rho \downarrow 0$ □

B_t d-dimensional Brownian motion starting at $x \in G$

$$\tau_G = \inf\{t \geq 0 \ : \ B(t) \notin G\}$$

$$u(x) = E_x[f(B(\tau_G))]$$

"Theorem" If ∂G "nice" then u solves the Dirichlet problem

$$E_x[f(B(\tau_G))] = \int_{\partial G} f(y)\pi_G(x, dy), \quad \pi_G(x, \Gamma) = P_x(B(\tau_G) \in \Gamma), \quad \Gamma \subset \partial G$$

Example. $G = B(x, r), \quad \pi_G(x, \Gamma) = \frac{|\Gamma|}{|\partial S(x,r)|}, \qquad \Gamma \subset S(x, r)$

Brownian motion is invariant under rotations
$\therefore \pi_G(x, \cdot)$ is invariant under rotations

Proposition

G bounded open $\subset \mathbb{R}^d$, f bounded measurable on ∂G. Then $u(x) = E_x[f(B(\tau_G))]$ is harmonic in G.

Proof.

$B = B(x,r) \subset G \qquad \tau_B \leq \tau_G$

Strong Markov property: $\quad u(B(\tau_S)) = E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]$

$$\begin{aligned} u(x) = E_x[f(B(\tau_G))] &= E_x[E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]] \\ &= E_x[u(B(\tau_S))] \\ &= \int_{\partial S} u(y)\pi_S(x,dy) \\ &= \frac{1}{|\partial S|}\int_{\partial S} u(y)dS \end{aligned}$$

So u satisfies the mean value property in G. □

$$a \in \partial G$$

To complete the proof that u solves the Dirichlet problem we need

$$\lim_{x\to a,\ x\in G} E_x[f(B(\tau_G))] = f(a) \qquad \text{It is not always true!}$$

Proposition

If $\lim_{\substack{x\to a \\ x\in G}} P_x[\tau_G > \epsilon] = 0$, $\forall \epsilon > 0$ then for any bdd mble function $f : \partial G \to \mathbb{R}$ which is continuous at a, $\lim_{\substack{x\to a \\ x\in G}} E_x[f(B(\tau_G))] = f(a)$

Proof.

Need: $\lim_{x\to a,\ x\in G} P_x\left(|B(\tau_G) - x| < \delta\right) = 1$

$$\begin{aligned} P_x(|B(\tau_G) - x| < \delta) &\geq P_x(\sup_{0\leq t\leq \epsilon} |B(t) - x| < \delta,\ \tau_G \leq \epsilon) \\ &\geq P_x(\sup_{0\leq t\leq \epsilon} |B(t) - x| < \delta) - P_x(\tau_G \leq \epsilon) \\ &\to 1 \ \text{ as } \ x \to a, x \in G \ \text{ then } \ \epsilon \downarrow 0 \end{aligned}$$

Proposition

$a \in \partial G$ is *regular* if $P_a(\sigma_G = 0) = 1 \quad \sigma_G = \inf\{t > 0 \ : \ B(t) \notin G\}$
a regular $\Leftrightarrow \quad \lim_{x\to a,\ x\in G} E_x[f(B_{\tau_G})] = f(a)\ \forall f$ bdd mble, cont at a

Proof of $\Rightarrow$

Enough to prove $P_x(\sigma_G < \epsilon)$ lower semi-continuous in x
Then $\limsup_{\substack{x\to a\\ x\in G}} P_x(\sigma_G < \epsilon) \geq P_a(\sigma_G < \epsilon) = 1$ and $\sigma_G \geq \tau_G$
But $\int p(0, x, \delta, y) P_y(\exists s \in (0, \epsilon - \delta), B(s) \notin G)$ continuous
and $\uparrow P_x(\sigma_G < \epsilon)$ as $\delta \downarrow 0$

Examples

1. If ∂G is a smooth manifold near a then a is regular by LIL
2. If $\exists$ cone C of height $h > 0$ and vertex at a such that $C - \{a\} \subset \bar{G}^C$ then a is a regular (exterior cone condition)
3. $d \leq 2$ always,$d \geq 3$ $\exists$counterexamples

Application to recurrence/transience of Brownian motion

$$G = \{y \in \mathbf{R}^d \;:\; \delta < |y| < R\} \qquad f = \begin{cases} 0 & |y| = R \\ 1 & |y| = \delta \end{cases}$$

$$u(x) = E_x[f(B(\tau_G))] = P_x(\tau_\delta < \tau_R) = \begin{cases} \frac{\log R - \log |x|}{\log R - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} & d > 2 \end{cases}$$

Theorem In $d \geq 2$, Brownian motion does not visit a point

Proof.

$$P_x(\tau_0 < \tau_R) = \lim_{\delta \downarrow 0} P_x(\tau_\delta < \tau_R) = \lim_{\delta \downarrow 0} \frac{\log R - \log |x|}{\log R - \log \delta} = 0$$

□

Theorem In $d = 2$, Brownian motion is recurrent, ie. comes arbitrarily close to any point arbitrarily many times

Proof.

$$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} P_x(\tau_\delta < \tau_R) = \lim_{R\uparrow\infty} \frac{\log R - \log|x|}{\log R - \log\delta} = 1$$

□

Theorem In $d \geq 3$, Brownian motion wanders off to infinity

Proof.

$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} = \left(\frac{|x|}{\delta}\right)^{2-d}$ if $|x| > \delta$, 0 otherwise

$P_x(\text{hit } |y| = \delta \text{ after time } t) = \int \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} P_y(\tau_\delta < \infty)dy \to 0$ as $t \to \infty$

$P_x(\liminf_{t\to\infty} |B(t)| > \delta) = 1$

$\delta \uparrow \infty \liminf_{t\to\infty} |B(t)| = \infty \qquad a.s.$

□

Strong Markov Property of solution of SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \qquad X_0 = x$$

τ a stopping time

$$f(X_t) - \int_0^t Lf(X_u)du = martingale$$

$$E[f(X_{t+\tau}) - \int_0^t Lf(X_{u+\tau})du \mid \mathcal{F}_{\tau+s}]$$
$$= f(X_{s+\tau}) - \int_0^s Lf(X_{u+\tau})du \qquad \text{optional stopping}$$

$\Rightarrow \tilde{X}_t = X_{\tau+t}$, $t \geq 0$ is a solution of

$$d\tilde{X}_t = b(t+\tau, \tilde{X}_t)dt + \sigma(t+\tau, \tilde{X}_t)d\tilde{B}_t \qquad t \geq 0$$

Generalized Dirichlet problem.

Let D be a domain in $\mathbb{R}^d$, i.e. a bounded connected open set with a smooth boundary ∂D

Suppose

$$\begin{cases} Lu = 0 & in\ D, \\ u = f & on\ \partial D \end{cases}$$

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i}$$

Let X_t be the solution of the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \qquad X_0 = x.$$

Let τ be the exit time from the region D

Then X_τ is the exit point on the boundary ∂D

$$u(x) = E_x[f(X_\tau)]$$

To show it we cannot use the same conditioning argument as with Brownian motion because we don't have the symmetry anymore

Ito's formula:

$$u(X_{t\wedge\tau}) = \textit{ martingale}$$

Optional stopping

$$E_x[f(X_\tau)] = E_x[u(X_\tau)] = u(x)$$

Poisson equation

Suppose

$$\begin{cases} Lu = g & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases}$$

where g is some given function defined in D

Ito's formula:

$$u(X_{t\wedge\tau}) - \int_0^{t\wedge\tau} g(X_s)ds = \text{ martingale}$$

Optional stopping: $E_x[u(X_\tau) - u(X_0) - \int_0^\tau g(X_s)ds] = 0.$

$$u(x) = E_x[\int_0^\tau g(X_s)ds]$$

Generalized Feynman-Kac formula

$$-\frac{\partial u}{\partial t} = \mathcal{L}u + Vu + g, \qquad u(T, x) = f(x)$$

$$u(t,x) = E_{t,x}[e^{\int_t^T V(s,X(s))ds} u_0(X(T)) + \int_t^T g(s, X(s)) e^{\int_t^s V(u,X(u))du} ds]$$

Proof.

Apply Itô's formula to

$$u(s, X(s)) e^{\int_t^s V(u,X(u))du}$$

□

Diffusion process as probability measure on $C([0,\infty))$

Brownian motion=collection of rv's B_t, $t \geq 0$ on $(\Omega, \mathcal{F}, P)$

or

Brownian motion=probability measure P on $C([0,\infty))$

If $\mathcal{A}$ is a (measurable) subset of continuous functions, then $P(\mathcal{A})$ is just the probability that a Brownian path falls in that subset

Same for any diffusion.If $X(t)$ is the solution of the stochastic differential equation $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$, $X(0) = x$ then we can let $P_x^{a,b}$ denote the probability measure on the space of continuous functions with

$$P_x^{a,b}(\mathcal{A}) = Prob(X(\cdot) \in \mathcal{A}) \qquad a = \sigma\sigma^T$$

Question:What is the relation of $P_x^{a,b}$ for different x, a, b?

1. If $x_1 \neq x_2$ then $P^{a,b}_{x_1} \perp P^{a,b}_{x_2}$.
2. The quadratic variation

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor 2^n T \rfloor} |X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})|^2 = \int_0^T a(t, X(t))dt \qquad \text{a.s.} \quad P^{a,b}_x$$

   Hence if $a_1 \neq a_2$, $P^{a_1,b}_x \perp P^{a_2,b}_x$

3. To see what happens if we change b, let $dX_i(t) = b_i dt + \sigma dB(t)$, $i = 1, 2$

$$\frac{P(X_1(t_1) \in dx_1, \dots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \dots, X_2(t_n) \in dx_n)}$$
$$= e^{-\sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i-b_1(t_{i+1}-t_i))^2-(x_{i+1}-x_i-b_2(t_{i+1}-t_i))^2}{2\sigma^2(t_{i+1}-t_i)}}$$
$$= e^Z dx_1 \cdots dx_n$$

$$\frac{P(X_1(t_1) \in dx_1, \ldots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \ldots, X_2(t_n) \in dx_n)} = e^Z dx_1 \cdots dx_n$$

$$\begin{aligned}
Z &= -\sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i - b_1(t_{i+1} - t_i))^2 - (x_{i+1} - x_i - b_2(t_{i+1} - t_i))^2}{2\sigma^2(t_{i+1} - t_i)} \\
&= -\sum_{i=0}^{n-1} \sigma^{-1}(b_2 - b_1)\sigma^{-1}(x_{i+1} - x_i - b_2(t_{i+1} - t_i)) \\
&\quad -\frac{1}{2}\sum_{i=0}^{n-1}(\sigma^{-1}(b_2 - b_1))^2(t_{i+1} - t_i) \\
&\stackrel{n\to\infty}{\longrightarrow} -\int_0^t \sigma^{-1}(b_2 - b_1)dB(s) - \frac{1}{2}\int_0^t (\sigma^{-1}(b_2 - b_1))^2 ds
\end{aligned}$$

Cameron-Martin-Girsanov formula

For each x the measure $P_x^{a,b}$ is absolutely continuous on $\mathcal{F}_t$ with respect to the measure $P_x^{a,0}$ and

$$\left.\frac{dP_x^{a,b}}{dP_x^{a,0}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t a^{-1}(X_s)b(X_s)dX_s - \frac{1}{2}\int_0^t b(X_s)a^{-1}(X_s)b(X_s)ds\right\}$$

Proof

We want to show is if we define a measure

$$Q(A) = \int_A \exp\left\{\int_0^t a^{-1}bdX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds\right\} dP_{x_0}^{a,0}$$

then Q is a diffusion with parameters a and b in other words, for each λ,

$$\exp\{\lambda(X_t - X_0 - \int bds) - \frac{\lambda^2}{2}\int_0^t ads\}$$

is a martingale with respect to Q

Proof.

$$Y_t = \int_0^t (\lambda + a^{-1}b)dX_s = \int_0^t (\lambda + a^{-1}b)\sigma dB_s.$$

$$e^{Y_t - Y_0 - \frac{1}{2}\int_0^t a(\lambda + a^{-1}b)^2 ds} = \text{martingale w.r.t. } P^{a,0}_{x_0}$$

$$e^{\lambda(X_t - X_0 - \int_0^t b ds) - \frac{\lambda^2}{2}\int_0^t a ds + \int_0^t a^{-1}b dX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds} = \text{martingale w.r.t. } P^{a,0}_{x_0}$$

$$e^{\lambda(X_t - X_0 - \int_0^t b ds) - \frac{\lambda^2}{2}\int_0^t a ds} = \text{martingale w.r.t. } Q$$

$$dQ = e^{\int_0^t a^{-1}b dX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds} dP^{a,0}_{x_0}$$

□