Martingale Transform

M_n martingale with respect to $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$ $\sigma_n \in \mathcal{F}_n$
$(\sigma \cdot M)_n = \sum_{i=0}^{n-1} \sigma_i (M_{i+1} - M_i)$ is a Martingale

$$\begin{aligned}
&E[(\sigma \cdot M)_n \mid \mathcal{F}_{n-1}] \\
&= E[\sum_{i=0}^{n-1} \sigma_i (M_{i+1} - M_i) \mid \mathcal{F}_{n-1}] \\
&= \sum_{i=0}^{n-2} \sigma_i (M_{i+1} - M_i) + \sigma_{n-1} E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}] \\
&= (\sigma \cdot M)_{n-1}
\end{aligned}$$

Called Martingale Transform "Fortune at time n using strategy σ "
Stochastic Integral $\int_0^t \sigma dB$ is continuous time version based on martingale $B(t)$.

Note one can also define stochastic integrals with respect to continuous martingales $M(t)$ other than Brownian motion. The only thing that changes is that we have the quadratic variation $\langle M, M \rangle_t = \lim \sum (M(t_{i+1}) - M(t_i))^2$.

Itô's Lemma

Let $f(x)$ be twice continuously differentiable. Then

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt$$

Proof

First of all we can assume without loss of generality that f, f' and f'' are all uniformly bounded , for if we can establish the lemma in the uniformly bounded case, we can approximate f by f_n so that all the corresponding derivatives are bounded and converge to those of f uniformly on compact sets.

Let $s = t_0 < t_1 < t_2 < \cdots < t_n = t$. We have

$$
\begin{aligned}
f(B(t)) - f(B(s)) &= \sum_{j=0}^{n-1} [f(B(t_{j+1})) - f(B(t_j))] \\
&= \sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \\
\xi_j \in [t_j, t_{j+1}] \qquad & \quad + \sum_{j=0}^{n-1} \frac{1}{2} f''(B(\xi_j))(B(t_{j+1}) - B(t_j))^2,
\end{aligned}
$$

Let the width of the partition go to zero. By definition of the stochastic integral

$$
\sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_s^t f' dB.
$$

Finally we want to show that

$$\sum_{j=0}^{n-1} f''(B(\xi_j))(B(t_{j+1}) - B(t_j))^2 \to \int_s^t f''(B(u))du$$

It's a lot like the computation of the quadratic variation.
Suppose we had $f''(B(t_j)$ inside instead of $f''(B(\xi_j))$

$$E\left[\left(\sum_{j=0}^{n-1} f''(B(t_j))\left[(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)\right]\right)^2\right]$$

$$= \sum_{i,j=0}^{n-1} E\left[f''(B(t_i))X_i f''(B(t_j))X_j\right] \qquad X_j = (B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)$$

Suppose $i < j$

$$\begin{aligned} E\left[f''(B(t_i))X_i f''(B(t_j))X_j\right] &= E\left[E\left[f''(B(t_i))X_i f''(B(t_j))X_j \mid \mathcal{F}(t_j)\right]\right] \\ &= E\left[f''(B(t_i))X_i f''(B(t_j))E\left[X_j \mid \mathcal{F}(t_j)\right]\right] = 0 \end{aligned}$$

Hence

$$E\left[\left(\sum_{j=0}^{n-1} f''(B(t_j))\left[(B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right]\right)^2\right]$$

$$=\sum_{j=0}^{n-1} E\left[(f''(B(t_i))^2\left[(B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right]^2\right]\to 0$$

so

$$\sum_{j=0}^{n-1} f''(B(t_j))(B(t_{j+1})-B(t_j))^2 \xrightarrow{L^2} \int_s^t f''(B(u))du$$

We still have to show $\sum_{j=0}^{n-1} |f''(B(\xi_j)) - f''(B(t_j)|(B(t_{j+1}) - B(t_j))^2 \to 0$
Taking expectation we get

$$\sum_{j=0}^{n-1} E[|f''(B(\xi_j)) - f''(B(t_j)|(B(t_{j+1}) - B(t_j))^2]$$

$$\leq \sum_{j=0}^{n-1} \sqrt{E[(f''(B(\xi_j)) - f''(B(t_j))^2]} \sqrt{E[(B(t_{j+1}) - B(t_j))^4]}$$

$$= C \sum_{j=0}^{n-1} \sqrt{E[(f''(B(\xi_j)) - f''(B(t_j))^2]} (t_{j+1} - t_j) \to 0$$

by continuity
So we have proved that

$$f(B(t) - f(B(s)) = \int_s^t f'(B(u))dB(u) + \frac{1}{2} \int_s^t f''(B(u))du$$

which is Itô's formula.

1. In differential notation Itô's formula reads

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt.$$

The Taylor series is $df(B) = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(B)(dB)^n$. In normal calculus we would have $(dB)^n = 0$ if $n \geq 2$, but because of the finite quadratic variation of Brownian paths we have $(dB)^2 = dt$, while still $(dB)^n = 0$ if $n \geq 3$.

2. If the function f depends on t as well as $B(t)$, the formula is

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \frac{\partial f}{\partial x}(t, B(t))dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B(t))dt.$$

The proof is about the same as the special case above.

Exponential Martingales

$$M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$$

Ito's formula

$$df(t,B(t)) = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})(t,B(t))dt + \frac{\partial f}{\partial x}(t,B(t))dB(t)$$

$f(t,x) = e^{\lambda x - \frac{1}{2}\lambda^2 t}$ $\quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial f}{\partial x} = \lambda f$

$$M_t = \int_0^t \lambda e^{\lambda B_s - \frac{1}{2}\lambda^2 s} dB_s$$

Geometric Brownian motion

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

is called Geometric (or Exponential) Brownian Motion
$S_t \geq 0$ so it is (Samuelson) a better model of stock prices than B_t (Bachelier)
$\mu = \text{drift} \quad \sigma = \text{volatility}$
Ito's formula

$$df(t, B(t)) = (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2})(t, B(t))dt + \frac{\partial f}{\partial x}(t, B(t))dB(t)$$

$f(t, x) = e^{(\mu - \frac{\sigma^2}{2})t + \sigma x} \quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu f \quad \frac{\partial f}{\partial x} = \sigma f$

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

Sometimes people write $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ but note that $\frac{dS_t}{S_t} \neq d \log S_t$

Local time

f continuous function on $\mathbb{R}_+$

$$L_t(x) = \int_0^t \delta_x(f(s))ds = \lim_{\epsilon \downarrow 0}(2\epsilon)^{-1}|\{0 \leq s \leq t : |f(s) - x| \leq \epsilon\}|$$

$$\int_0^t 1_A(f(s))ds = \int_A L_t(x)dx$$

$f \in C^1$ $L_t(x) = \sum_{s_i \in [0,t]: f(s_i) = x} |f'(s_i)|^{-1}$ discontinuous in t
Itô 's lemma applied to $|B_t - x|$ gives

Tanaka's formula for Brownian Local Time

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \mathrm{sgn}(B_s - x)dB_s$$

In particular, $L_t(x)$ continuous in t a.s.

But $|x|$ not bounded, so no fair!!!

Proof.

$$f''_\epsilon(x) = (2\epsilon)^{-1} 1_{[-\epsilon,\epsilon]}$$

Itô

$$(2\epsilon)^{-1}|\{0 \le s \le t : |B_s| \le \epsilon\}| = f_\epsilon(B_t) - f_\epsilon(B_0) - \int_0^t f'_\epsilon(B_s)dB_s$$

$$\epsilon \downarrow 0$$

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \mathrm{sgn}(B_s - x)dB_s$$

□

Feynman-Kac formula

V a nice function (say bounded). $u \in C^{1,2}$ solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu, \qquad u(0,x) = u_0(x)$$

$\int u(x)\exp\{-x^2/2t\}dx < \infty$. Then

$$u(t,x) = E_x\left[e^{\int_0^t V(B(s))ds} u_0(B(t))\right]$$

Proof.

For $0 \le s \le t$ let $Z(s) = u(t-s, B(s))e^{\int_0^s V(B(u))du}$. By Îto's lemma

$$\begin{aligned} Z(t) - Z(0) &= \int_0^t \left\{-\frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu\right\} e^{\int_0^s V(B(u))du} ds \qquad = 0 \\ &+ \int_0^t \frac{\partial u}{\partial x}(t-s, B(s))e^{\int_0^s V(B(u))du} ds \qquad = \text{martingale} \end{aligned}$$

so $E_x[Z(t)] = E_x[Z(0)]$ □

1. If $V = V(t,x)$

$$u(t,x) = E_x\left[e^{\int_0^t V(t-s,B(s))ds} u_0(B(t))\right]$$

2. Historical remark. Feynman's thesis was that solution u of it Schrödinger equation $\frac{\partial u}{\partial t} = i[\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu]$ should have representation

$$u(x,t) = \text{''}\int\text{''}\, e^{i\int_0^t V(f_s)ds - \frac{i}{2}\int_0^t |f'|^2 ds} u_0(f_t)$$

where " $\int$ " is supposed to be average over functions starting at x. Kac pointed out that it is rigorous if $i \mapsto 1$

Arcsin law

$\xi(t) = \frac{1}{t}\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds =$ the fraction of time that Brownian motion is positive up to time t

$$P(\xi(t) \le a) = \begin{cases} 0 & a < 0; \\ \frac{2}{\pi}\arcsin\sqrt{a} & 0 \le a \le 1; \\ 1 & a > 1. \end{cases}$$

Simple explanation why distribution of $\xi(t)$ indep of t

$$\xi(t) = \int_0^1 \mathbf{1}_{[0,\infty)}(B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\frac{1}{\sqrt{t}}B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\tilde{B}(s))ds$$

$$\xi(t) \stackrel{d}{=} \xi(1)$$

Proof

By Feynman-Kac if we can find a nice solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u\mathbf{1}_{[0,\infty)} \qquad\qquad u(0,x) = 1$$

Then

$$u(t,x) = E_x[e^{-\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds}]$$

and

$$u(t,0) = \int_0^1 e^{-at} dP(\xi \le a)$$

$\alpha > 0$, $\phi_\alpha(x) = \alpha\int_0^\infty u(t,x)e^{-\alpha t}dt \longrightarrow -\frac{1}{2}\phi_\alpha'' + (\alpha + \mathbf{1}_{[0,\infty)})\phi_\alpha = \alpha$

$$\phi_\alpha(x) = \begin{cases} \frac{\alpha}{\alpha+1} + Ae^{x\sqrt{2(\alpha+1)}} + Be^{-x\sqrt{2(\alpha+1)}}, & x \ge 0, \\ 1 + Ce^{x\sqrt{2\alpha}} + De^{-x\sqrt{2\alpha}}, & x \ge 0. \end{cases}$$

$u \le 1 \Rightarrow \phi_\alpha \le 1 \Rightarrow A = D = 0$

Proof.

$\phi_\alpha(0_-) = \phi_\alpha(0_+), \phi'_\alpha(0_-) = \phi'_\alpha(0_+)$

$\Rightarrow B = \frac{\alpha^{1/2}}{(1+\alpha)(\sqrt{\alpha}+\sqrt{\alpha+1})}, C = \frac{1}{(1+\alpha)^{1/2}(\sqrt{\alpha}+\sqrt{\alpha+1})^{1/2}}$

$$\phi_\alpha(0) = \sqrt{\frac{\alpha}{\alpha+1}} = \int_0^\infty E[e^{-t\xi}\alpha e^{-\alpha t}]dt$$

By Fubini's theorem this reads $E\left[\frac{\alpha}{\alpha+\xi}\right] = \sqrt{\frac{\alpha}{\alpha+1}}$or

$$\int_0^1 \frac{1}{1+\gamma a} dP(\xi \leq a) = \frac{1}{\sqrt{1+\gamma}}$$

Looking up a table of transforms we find

$$dP(\xi \leq a) = \frac{2}{\pi} \frac{1}{\sqrt{a(1-a)}} da \qquad 0 \leq a \leq 1$$

which is the density of the arcsin distribution □

Stochastic differential equations

$\sigma(x,t)$, $b(x,t)$ mble

Definition

A stochastic process X_t is a solution of a stochastic differential equation

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

on $[0,T]$ if X_t is progressively measurable with respect to $\mathcal{F}_t$, $\int_0^T |b(X_t,t)|dt < \infty$, $\int_0^T |\sigma(X_t,t)|^2 dt < \infty$ a.s. and

$$X_t = x_0 + \int_0^t b(X_s,s)ds + \int_0^T \sigma(X_s,s)dB_s \qquad 0 \le t \le T$$

The main point is that $\sigma(\omega,t) = \sigma(X_t,t)$, $b(\omega,t) = b(X_t,t)$

If $B(t)$ is a d-dimensional Brownian motion and $f(t,x)$ is a function on $[0,\infty) \times \mathbf{R}^d$ which has one continuous derivative in t and two continuous derivatives in x, then Ito's formula reads

$$df(t,B(t)) = \frac{\partial f}{\partial t}(t,B(t))dt + \nabla f((t,B(t)) \cdot dB(t) + \frac{1}{2}\Delta f(t,B(t))dt.$$

Bessel process ($d = 2$)

Let $B_t = (B_t^1, B_t^2)$ be 2d Brownian motion starting at 0,

$$r_t = |B_t| = \sqrt{(B_t^1)^2 + (B_t^2)^2}.$$

By Ito's lemma,

$$dr_t = \frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2 + \frac{1}{2} \frac{1}{|B|} dt.$$

This is *not* a stochastic differential equation.

$$Y(t) = \int_0^t \frac{B^1}{|B|} dB^1 + \int_0^t \frac{B^2}{|B|} dB^2$$

Let $f(t, y)$ be a smooth function Use Itô's lemma. Intuitively

$$df(t, Y_t) = \partial_t f dt + \partial_y f dY + \frac{1}{2}\partial_y^2 f (dY)^2$$

$$\begin{aligned} (dY)^2 &= (\frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2)^2 \\ &= \left(\frac{B^1}{|B|}\right)^2 (dB^1)^2 + 2\frac{B^1 B^2}{|B|^2} dB^1 dB^2 + \left(\frac{B^2}{|B|}\right)^2 (dB^2)^2 \\ &= dt \end{aligned}$$

$$\begin{aligned} f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s) ds \\ &\quad + \int_0^t \partial_y f \frac{B^1}{|B|} dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|} dB^2 \end{aligned}$$

Itô's lemma

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \Big\{\partial_s f(s, X_s) + \mathcal{L}f(s, X_s)\Big\}ds \\ &\quad + \int_0^t \sum_{i,j=1}^d \sigma_{ij}(s, X_s)\frac{\partial}{\partial x_i}f(s, X_s)dB_s^j \end{aligned}$$

$$\mathcal{L}f(t, x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(t, x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^d b_i(t, x)\frac{\partial f}{\partial x_i}(t, x)$$

$$a_{ij} = \sum_{k=1}^d \sigma_{ik}\sigma_{jk} \qquad a = \sigma\sigma^T$$

$$dr_t = \frac{B^1}{|B|}dB^1 + \frac{B^2}{|B|}dB^2 + \frac{1}{2}\frac{1}{|B|}dt = dY_t + \frac{1}{2}r_t^{-1}dt$$

$$\begin{aligned} f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s)ds \\ &\quad + \int_0^t \partial_y f \frac{B^1}{|B|}dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|}dB^2 \end{aligned}$$

In particular $e^{\lambda Y_t - \lambda^2 t/2}$ is a martingale

So Y_t is a Brownian motion.

Therefore

$$dr_t = dY_t + \frac{1}{2}r_t^{-1}dt$$

is a stochastic differential equation for the new Brownian motion Y_t

Itô's lemma

$$f(t, X_t) - f(0, X_0) = \int_0^t \left\{\partial_s + \mathcal{L}\right\} f(s, X_s) ds + \int_0^t \nabla f(s, X_s) \cdot \sigma dB_s$$

Proof

$$= \sum_i f(t_{i+1}, X_{t_{i+1}}) - f(t_i, X_{t_i})$$

$$= \sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) + \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})$$

$$+ \frac{1}{2} \sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i})$$

$$+ \text{ higher order terms}$$

Proof continued

$$\sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) \rightarrow \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds$$

$$\sum_i \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})$$

$$= \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} \sigma(s, X_s)dB_s) \quad \rightarrow \int_0^t \nabla f \cdot \sigma dB$$

$$+ \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} b(s, X_s)ds) \quad \rightarrow \int_0^t \nabla f \cdot b\, ds$$

$$\sum_i \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \rightarrow \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(s, X_s)a_{jk}(s, X_s)ds$$

Proof continued

To show the last convergence, ie

$$\sum_i g(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \to \int_0^t g(s, X_s) a_{jk}(s, X_s) ds$$

$$\begin{aligned} Z(t_i, t_{i+1}) &= \left(\int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}(s, X_s) dB^l_s\right)\left(\int_{t_i}^{t_{i+1}} \sum_m \sigma_{km}(s, X_s) dB^m_s\right) \\ &\quad - \int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}\sigma_{kl}(s, X_s) ds \end{aligned}$$

$$E[|Z(t_i, t_{i+1})|^2] = \mathcal{O}((t_{i+1} - t_i)^2)$$

Proof continued

$$\sum_i g(t_i, X_{t_i}) E[\int_{t_i}^{t_{i+1}} a_{ij}(s, X_s) ds] \to \int_0^t g(s, X_s) a_{ij}(s, X_s) ds$$

$$E[(\sum_i g(t_i, X_{t_i}) Z(t_i, t_{i+1}))^2] = \sum_{i,j} E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1})]$$

$$i < j \quad E[E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1}) \mid \mathcal{F}_{t_j}]] = 0$$

$$\begin{aligned} i = j \qquad & E[E[g^2(t_i, X_{t_i}) Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = E[g^2(t_i, X_{t_i}) E[Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = \mathcal{O}((t_{i+1} - t_i)^2) \end{aligned}$$

$$f(t,X_t) - f(0,X_0) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds = \int_0^t \nabla f(s,X_s)\cdot\sigma dB_s$$

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i} \quad = \textit{generator}$$

$M_t = f(t,X_t) - \int_0^t \Big\{\partial_s + \mathcal{L}\Big\} f(s,X_s)ds$ is a martingale

$$\begin{aligned} 0 &= E[f(t,X_t) - f(s,X_s) - \int_s^t \Big\{\partial_u + \mathcal{L}\Big\} f(u,X_u)du \mid \mathcal{F}_s] \\ &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \Big\{\partial_u + \mathcal{L}\Big\} f(u,y)p(s,x,u,y)dydu, \qquad X_s = x \end{aligned}$$

For any f,

$$\begin{aligned} 0 &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\ &\quad - \int_s^t \int \left\{\partial_u + \mathcal{L}\right\} f(u,y)p(s,x,u,y)dydu \end{aligned}$$

Fokker-Planck (Forward) Equation

$$\begin{aligned} \frac{\partial}{\partial t} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}(a_{i,j}(t,y)p(s,x,t,y)) \\ &\quad - \sum_{i=1}^{d} \frac{\partial}{\partial y_i}(b_i(t,y)p(s,x,t,y)) \\ &= L_y^* p(s,x,t,y) \end{aligned}$$

$$\lim_{t \downarrow s} p(s,x,t,y) = \delta(y-x).$$

Kolmogorov (Backward) Equation

$$
\begin{aligned}
-\frac{\partial}{\partial s} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\frac{\partial^2 p(s,x,t,y)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{d} b_i(s,x)\frac{\partial p(s,x,t,y)}{\partial x_i} \\
&= L_x p(s,x,t,y)
\end{aligned}
$$

$$\lim_{s \uparrow t} p(s,x,t,y) = \delta(y-x).$$

Proof.

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \leq s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time t

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,z)dz$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$. We get in the limit that p satisfy the backward equations. □

Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y), \quad t > s,$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$