PROBLEMS (due Mar 15)

1. The *nth Hermite polynomial* is $H_n(t,x) = \frac{(-t)^n}{n!} e^{\frac{x^2}{2t}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2t}}$. Show that the H_n play the role that the monomials $\frac{x^n}{n}$ play in ordinary calculus,

$$dH_{n+1}(t,B_t) = H_n(t,B_t).$$

2. The backward equation for the Ornstein-Uhlenbeck process is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \rho x \frac{\partial u}{\partial x}.$$

Show that

$$v(t,x) = u(t, xe^{\rho t})$$

satisfies

$$e^{2\rho t}\frac{\partial v}{\partial t} = \frac{1}{2}\frac{\partial^2 v}{\partial x^2}$$

and transform this to the heat equation by $\tau = \frac{1-e^{-2\rho t}}{2\rho}$. Use this to derive *Mehler's formula* for the transition probabilities,

$$p(t,x,dy) = \frac{e^{-\frac{\rho(y-xe^{-\rho t})^2}{(1-e^{-2\rho t})}}}{\sqrt{2\pi\left(\frac{(1-e^{-2\rho t})}{2\rho}\right)}} dy.$$

3. i. Show that $X(t) = (1-t)\int_0^t \frac{1}{1-s} dB(s)$ is the solution of

$$dX(t) = -\frac{X(t)}{1-t}dt + dB(t), \qquad 0 \le t < 1, \qquad X(0) = 0.$$

ii. Show that $X(t)$ is Gaussian and find the mean and covariance.

iii. Show that for $0 = t_0 < t_1 < \cdots < t_n < 1$ the variables $\frac{X(t_i)}{1-t_i} - \frac{X(t_{i-1})}{1-t_{i-1}}$ are independent

iv. Show that the finite dimensional distributions are given by

$$\begin{aligned} &P(X(t_1) \in dx_1, \ldots, X(t_n) \in dx_n) \\ &= \prod_{i=1}^n p(t_i - t_{i-1}, x_i - x_{i-1}) \frac{p(1-t_n, -x_n)}{p(1,0)} dx_1 \cdots dx_n \end{aligned}$$

where $p(t,x)$ is the Gaussian kernel.

v. Show that $X(t)$ is equal in distribution to a Brownian motion conditioned to have $B(1) = 0$. It is the *Brownian Bridge*.

vi. For fixed constants a and b solve the stochastic differential equation

$$dX(t) = \frac{b - X(t)}{1-t} dt + dB(t), \qquad 0 \leq t < 1 \qquad X(0) = a.$$

This is the Brownian Bridge from a to b.

4. Consider the general linear stochastic differential equation

$$dX_t = [A(t)X_t + a(t)]dt + \sigma(t)dB_t, \qquad X_0 = x,$$

where B_t is an r-dimensional Brownian motion independent of the initial vector $x \in \mathbf{R^d}$ and the $d \times d$, $d \times 1$ and $d \times r$ matrices $A(t)$, $a(t)$ and $\sigma(t)$ are non-random. Show that the solution is given by

$$X_t = \Phi(t)[x + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s]$$

where Φ is the $d \times d$ matrix solution of $\dot{\Phi}(t) = A(t)\Phi(t), \qquad \Phi(0) = I.$