PROBLEMS (due Feb 8)

1. Let $B(t)$ be Brownian motion. Prove that $X = \int_0^1 B^2(s)ds$ is a random variable and compute the first two moments. Is X Gaussian?

2. Let $B(t)$ be Brownian motion. Prove that with probability one,

$$\lim_{n\to\infty} \sum_{i<2^n t} |B_{(i+1)2^{-n}} - B_{i2^{-n}}|^2 = t$$

(Hint: Compute the variance and use Borel-Cantelli)

3. Let $p \in (-1/2, 1/2)$. For each fixed y prove that $f_y(x) = |x-y|^{-p} - |x|^{-p}$ is in $L^2(\mathbf{R})$.

Let $X(f) = \int f dB$ be the Gaussian measure corresponding to Brownian motion, ie. the Gaussian measure corresponding to Lebesgue measure on $\mathbf{R}$. Let

$$Z_y = X(f_y)$$

Z_y is called *fractional Brownian motion* of index α. Find the distribution (ie. all finite dimensional distributions) of Z_y.

4. Let τ be a stopping time. Show that

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \ : \ A \cap \{\tau \le n\} \in \mathcal{F}_n, \ n \ge 0\}$$

is a σ-field.

5. If τ and σ are stopping times, then so are $\tau + \sigma$, $\max(\tau, \sigma)$ and $\min(\tau, \sigma)$.

6. Let B_t, $t \ge 0$ be Brownian motion. Use the law of the iterated logarithm,

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t \log\log t^{-1}}} = 1 \qquad a.s.$$

to find all subsequential limit points of

$$\frac{B_t}{\sqrt{t \log\log t}}$$

as $t \to \infty$.

7. Let $X_1, X_2, \ldots$ be iid with $E[X_i] = 0$ and $E[X_i^2] = 1$ and $S_n = X_1 + \cdots + X_n$. The law of the iterated logarithm also holds in this case

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1 \qquad a.s.$$

(if you are interested, it can be derived from the LIL for Brownian motion by embedding the S_n into the Brownian motion, see eg. Durrett)

On the other hand the central limit theorem says that $\frac{S_n}{\sqrt{n}}$ converges in distribution to a Gaussian.

Explain why the two results are not in contradiction.