PRACTICE PROBLEMS

1. Solve the stochastic differential equation $dX = XdB$, $X(0) = 1$. Does the solution ever become negative?

2. Consider the Ehrenfests model: An urn contains n balls. m of the balls are white, and the rest are black. At each time step a ball is chosen at random from the urn and replaced with one of the opposite colour. Let X n k be the number of white balls minus the number of black balls after k steps.

   (a) Write down the transition probabilities for the Markov chain.

   (b) Show that $Y_n(t) = n^{-1/2} X^n_{\lfloor n \rfloor}$ converges to the Ornstein-Uhlenbeck process.

3. Let V be the value of a European option where the interest rate is r and the underlying stock price process satisfies $dS = \mu S dt + \sigma S dB$. Let $\Gamma = \frac{\partial^2 V}{\partial S^2}$ and vega $v = \frac{\partial V}{\partial \sigma}$

   (a) Write a partial differential equation for vega as a function of S and t, in terms of σ, r and Γ

   (b) Find the correct boundary condition at the expiry date.

4. Find the forward and backward equations for the Bessel process $r(t) = \sqrt{|B_1(t)|^2 + |B_2(t)|^2}$ where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Include the boundary conditions.

5. Let M_t be a martingale under P with respect to $\mathcal{F}_\sqcup$ and a_t a continuous progressively measurable function of finite variation. Suppose both M_t and a_t are bounded on $[0, T]$. Show that

$$a_t M_t - \int_0^t M_s da_s$$

   is a martingale.

6. Let $f_t(x)dx = P(B_t \in dx \mid \sup_{0 \le s \le t} |B_s| < 1)$ be the density of Brownian motion conditioned to stay in $(-1, 1)$ up to time t. Compute $\lim_{t \to \infty} f_t(x)$.

7. Find the infinitesimal generator of the process $dx_1 = dt$, $dx_2 = dB$ which produces the graph of Brownian motion.

8. Suppose that B_t is a Brownian motion on the unit circle. Find a pde (with boundary conditions) for $E_x[arg(B_t)]$.

9. Let $G(x, y)$ be the Green function for $L = \frac{1}{2}\Delta + V$ on a nice bounded connected set $\Omega \subset \mathbf{R^d}$, ie. $Lh = f$ has solution $h(x) = \int G(x, y) f(y) dy$. Find a representation for $G(x, y)$ using Brownian motion.

10. Find the quadratic variation of the Feller diffusion $dX = \sqrt{X} dB$.

11. Let $X_t = B_t^2$ where B_t is a Brownian motion. Is X_t Markov?