Strong Markov property of Brownian motion

B_t, $t \geq 0$ be a Brownian motion with respect to $\mathcal{F}_t$, $t \geq 0$
τ a bounded stopping time.
$\tilde{B}_t = B_{t+\tau} - B_\tau$.
Then $\tilde{B}_t$ is a Brownian motion independent of $\mathcal{F}_\tau$.

In other words, Brownian motion starts afresh at every stopping time.

Proof.

optional stopping $+$ martingale characterization of Brownian motion □

Reflection Principle

For all $x > 0$, $t \geq 0$,

$$P(\sup_{0 \leq s \leq t} B_s \geq x) = 2P(B_t \geq x)$$

In particular $\sup_{0 \leq s \leq t} B_s \stackrel{d}{=} |B_t|$

Proof.

$\tau_x = \inf\{t \geq 0 : B_t \geq x\}$

$$P(\sup_{0 \leq s \leq t} B_s \geq x) = P(B_t \geq x) + P(\sup_{0 \leq s \leq t} B_s \geq x, B_t < x) = P(\tau_x \leq t)$$

$$P(\sup_{0 \leq s \leq t} B_s \geq x, B_t < x) = P(\tau_x \leq t, B_{\tau_x + (t - \tau_x)} - B_{\tau_x} < 0) = \frac{1}{2} P(\tau_x \leq t)$$

□

$$P(\tau_x \leq t) = 2\int_x^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy$$

Differentiate in t to see that τ_x has density

$$\begin{aligned} f_x(t) &= \frac{1}{\sqrt{2\pi}} \left(-\frac{1}{t^{3/2}} \int_x^\infty e^{-\frac{y^2}{2t}} dy + \frac{1}{t^{5/2}} \int_x^\infty y^2 e^{-\frac{y^2}{2t}} dy \right) \\ &= \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}} \end{aligned}$$

In particular $E[\tau_x] = \infty$

$$\tau_x = \inf\{t \geq 0 : x^{-1} B_t = 1\} = \inf\{x^2 t : \tilde{B}_t = x^{-1} B_{x^2 t} = 1\} = x^2 \tilde{\tau}_1$$

Note τ_x has stationary, independent increments, i.e. it is a Lévy process

Law of the iterated logarithm (Khinchine)

B_t, $t \geq 0$ Brownian motion

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log t^{-1}}} = 1 \qquad a.s.$$

Proof

To simplify expressions let $h(t) = \sqrt{2t\log\log t^{-1}}$
Step 1. Is to show $\limsup_{t\to 0} \frac{B_t}{h(t)} \leq 1$ a.s.
Applying Doob's inequality to the martingale $\exp\{aX_t - a^2t/2\}$

$$P(\sup_{0\leq s\leq t} \{B_s - as/2\} \geq \lambda) \leq e^{-a\lambda}$$

Let $\epsilon > 0 \quad 0 < \theta < 1 \quad t_n = \theta^n \;\; a_n = (1+\epsilon)\theta^{-n}h(\theta^n) \quad \lambda_n = h(\theta^n)/2$
Borel-Cantelli: $P(\sup_{0\leq s\leq t_n} B_s \geq a_n t_n/2 + \lambda_n \; i.o.) = 0$

Proof of LIL

Step 2. Is to show $\limsup_{t\to 0} \frac{B_t}{h(t)} \geq 1$ a.s.
Let $\epsilon > 0 \quad \theta = \epsilon^2/16$

$$A_n = \{B_{\theta^n} - B_{\theta^{n+1}} \geq (1-\sqrt{\theta})h(\theta^n)\} \qquad A_n \ \textit{independent}$$

If $\sum P(A_n) = \infty$ by Borel-Cantelli lemma $P(A_n\ i.o.) = 1$
i.e. with probability one there are infinitely many n for which

$$B_{\theta^n} \geq (1-\sqrt{\theta})h(\theta^n) + B_{\theta^{n+1}}$$

Step 1 $\Rightarrow \forall n \geq N,\ B_{\theta^{n+1}} < 2h(\theta^{n+1}) \overset{symmetry}{\Rightarrow} B_{\theta^{n+1}} > -2h(\theta^{n+1})$
so $B_{\theta^n} \geq (1-\sqrt{\theta})h(\theta^n) - 2h(\theta^{n+1}) \geq (1-\epsilon)h(\theta^n)$
so $\limsup_{t\to 0} B_t/h(t) \geq (1-\epsilon)$
Hence it suffices to prove that $\sum P(A_n) = \infty$

Proof of LIL

$$P(B_{\theta^n} - B_{\theta^{n+1}} \geq (1-\sqrt{\theta})h(\theta^n)) = \int_{\frac{(1-\sqrt{\theta})h(\theta^n)}{\sqrt{\theta^n-\theta^{n+1}}}}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

$$\geq C(\log n)^{-1/2} n^{-\frac{(1-\sqrt{\theta})^2}{1-\theta}}$$

by

$$\int_a^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx \geq \frac{1}{a+a^{-1}} \frac{e^{-\frac{a^2}{2}}}{\sqrt{2\pi}}$$

Since $(1-\sqrt{\theta})^2 = 1 - 2\sqrt{\theta} + \theta < 1 - \theta$, the series diverges QED

Prop

$|\{t : B_t = 0\}| = 0$ and $\{t : B_t = 0\}$ is perfect

Proof.

Recall *perfect* = *closed* and *any point is a limit point*
closed is obvious

$$E[\int_0^1 1_{\{t:B_t=0\}}(s)ds] = \int_0^1 P(0,0,s,\{0\})ds = 0 \quad \Rightarrow |\{t : B_t = 0\}| = 0$$

LIL $\Rightarrow$ 0 is a limit point of $\{t : B_t = 0\}$
$B_s = 0 \Rightarrow B_{t+s} - B_s$ Brownian motion so s is a limit point of $\{t : B_{t+s} - B_s = 0\}$

□

so $\{t : B_t = 0\}$ is like a Cantor set. $\{t : B_t = 0\} = \cup_{n=1}^{\infty} I_n$, I_n disjoint intervals
B_t, $t \in I_n$ is called a Brownian excursion

If f_t is a continuous function on $[0, \infty)$ and $\tau_x = \inf\{t : f_t \geq x\}$ then τ_x is continuous on $[a, b]$ iff $\sup_{0 \leq s \leq t} f_t$ is strictly increasing on $[\tau_a, \tau_b]$

Prop

$\{\tau_x\}_{x \geq 0}$ is not continuous in any interval

Proof.

LIL $\Rightarrow$

$$P(\tau_x \textit{ continuous on } [a, b]) = 0$$

Hence

$$P\Big(\bigcup_{a<b\in\mathbb{Q}} \{\tau_x \textit{ continuous on } [a, b]\}\Big) = 0$$

□

So $\{\tau_x\}_{x \geq 0}$ is a non-decreasing, discontinuous Lévy process

Another way to compute the distribution of τ_x, $x \geq 0$

$M_t = e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale

Optional stopping $\Rightarrow E[e^{\lambda B_{\min(\tau_x, B)} - \frac{1}{2}\lambda^2 \min(\tau_x, B)}] = 1$

BCT $\Rightarrow E[e^{\lambda B_{\tau_x} - \frac{1}{2}\lambda^2 \tau_x}] = 1, \qquad \lambda \geq 0$

so $E[e^{-\lambda \tau_x}] = e^{-\sqrt{2\lambda}x}, \qquad \lambda \geq 0$

In particular if τ_x and $\tilde{\tau}_y$ are independent then $\tau_x + \tilde{\tau}_y \overset{dist}{=} \tau_{x+y}$

Stable laws

X has stable distribution if for each n there is $0 < \alpha \leq 2$ and μ_n such that if $X_1, X_2, \ldots$ are iid with $X_i \overset{dist}{=} X$ then

$$\frac{X_1 + \cdots + X_n - \mu_n}{n^{1/\alpha}} \overset{dist}{=} X$$

Examples: Gaussian $\alpha = 2$; Cauchy $\alpha = 1$; τ_1 $\alpha = 1/2$

Subordinator means X_t non-decreasing (so it can be used as a time).

τ_x, $x \geq 0$ is a stable subordinator eg. $B_{\tau_x} = x$

$\tau = \inf\{t \geq 0 : B_t \textit{ hits } b + mt\}$, $b, m > 0$. What is $P(\tau < \infty)$?

$$E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)}] = 1$$

$B_{\min(\tau,N)} \leq b + m\min(\tau, N)$ so if $\lambda m \leq \frac{1}{2}\lambda^2$ we can let $N \to \infty$ to get

$$E[e^{\lambda(b+m\tau) - \frac{1}{2}\lambda^2\tau} 1_{\tau<\infty}] + \lim_{N\to\infty} E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] = 1$$

$$\begin{aligned} &\lim_{N\to\infty} E[e^{\lambda B_{\min(\tau,N)} - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] \\ &\leq \lim_{N\to\infty} E[e^{\lambda(b + m\min(\tau,N)) - \frac{1}{2}\lambda^2 \min(\tau,N)} 1_{\tau=\infty}] \\ &= 0 \end{aligned}$$

so

$$E[e^{(\lambda m - \frac{1}{2}\lambda^2)\tau} 1_{\tau<\infty}] = e^{-\lambda b}$$

Let $\lambda \downarrow 2m$ to get

$$P(\tau < \infty) = e^{-2mb}$$

Brownian motion starting at x: $B_t + x$, $B_0 = 0$

$P_x(A) = P(A \mid B_0 = x)$

$a < x < b$ What is $P_x(\tau_a < \tau_b)$?

If $B_0 = x$ then B_t is still a martingale

$E_x[B_{\min(\tau_a,\tau_b,N)}] = x$

$N \to \infty$, BCT$\Rightarrow$ $E_x[B_{\min(\tau_a,\tau_b)}] = x = E_x[B_{\tau_a} 1_{\tau_a < \tau_b}] + E_x[B_{\tau_b} 1_{\tau_a > \tau_b}]$
$= aP_x(\tau_a < \tau_b) + bP_x(\tau_a > \tau_b)$

$f(x) = P_x(\tau_a < \tau_b)$ $af(x) + b(1 - f(x)) = x$

$f(x) = P_x(\tau_a < \tau_b) = \frac{b-x}{b-a}$

Brownian motion in $\mathbb{R}^d$

1. $B_t = (B_t^1, \ldots, B_t^d)$, B_t^i independent Brownian motions
2. B_t Markov with $P(B_t \in A \mid B_s = x) = \int_A \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|y-x|^2}{2(t-s)}} dy$
3. B_t has stationary independent mean zero increments with $E[|B_t - B_s|^2] = d(t-s)$
4. $e^{\lambda \cdot B_t - \frac{1}{2}|\lambda|^2 t}$ is a martingale for any λ

Note that 1 does not depend on the basis: If $B_t^1, \ldots, B_t^2$ independent and $\mathcal{O}$ is orthogonal, then the coordinates of $\mathcal{O}B_t$ are independent Brownian motions   in fact

Theorem

Suppose X_1, X_2 independent and $\exists \theta \neq N\pi/2$ such that

$$X_1 \cos\theta + X_2 \sin\theta, \qquad -X_1 \sin\theta + X_2 \cos\theta \qquad \text{independent}$$

Then X_1, X_2 are Gaussians (Maxwell)

Dirichlet problem

Given a bounded open subset $G \subset \mathbf{R}^d$ and a continuous function $f : \partial G \to \mathbf{R}$ find a continuous function $u : \bar{G} \to R$ such that

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u|_{\partial G} = f \end{cases}$$

$$\Delta u \stackrel{def}{=} \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 2d \lim_{r \to 0} r^{-2} \left(\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS - u(x) \right)$$

Lemma

u harmonic in $G \Leftrightarrow u$ satisfies the mean value property: for all sufficiently small $r > 0$,

$$\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} u dS = u(x)$$

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Proof.

Green's identity $\int_G v \Delta u dx = \int_G u \Delta v dx + \int_{\partial G} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \; dS$

$$G = \{\delta < x < r\}, \qquad v = \begin{cases} \frac{\log r - \log |x|}{\log r - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - r^{2-d}}{\delta^{2-d} - r^{2-d}} & d > 2 \end{cases}$$

let $\rho \downarrow 0$ □

B_t d-dimensional Brownian motion starting at $x \in G$

$$\tau_G = \inf\{t \geq 0 \ : \ B(t) \notin G\}$$

$$u(x) = E_x[f(B(\tau_G))]$$

"Theorem" If ∂G "nice" then u solves the Dirichlet problem

$$E_x[f(B(\tau_G))] = \int_{\partial G} f(y)\pi_G(x, dy), \quad \pi_G(x, \Gamma) = P_x(B(\tau_G) \in \Gamma), \quad \Gamma \subset \partial G$$

Example. $G = B(x, r), \quad \pi_G(x, \Gamma) = \frac{|\Gamma|}{|\partial S(x,r)|}, \qquad \Gamma \subset S(x, r)$

Brownian motion is invariant under rotations
$\therefore \pi_G(x, \cdot)$ is invariant under rotations

Proposition

G bounded open $\subset \mathbb{R}^d$, f bounded measurable on ∂G. Then $u(x) = E_x[f(B(\tau_G))]$ is harmonic in G.

Proof.

$B = B(x,r) \subset G \qquad \tau_B \leq \tau_G$

Strong Markov property: $\quad u(B(\tau_S)) = E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]$

$$\begin{aligned} u(x) = E_x[f(B(\tau_G))] &= E_x[E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]] \\ &= E_x[u(B(\tau_S))] \\ &= \int_{\partial S} u(y)\pi_S(x,dy) \\ &= \frac{1}{|\partial S|}\int_{\partial S} u(y)dS \end{aligned}$$

So u satisfies the mean value property in G. □

$$a \in \partial G$$

To complete the proof that u solves the Dirichlet problem we need

$$\lim_{x \to a,\ x \in G} E_x[f(B(\tau_G))] = f(a) \qquad \text{It is not always true!}$$

Proposition

If $\lim_{\substack{x \to a \\ x \in G}} P_x[\tau_G > \epsilon] = 0$, $\forall \epsilon > 0$ then for any bdd mble function $f : \partial G \to \mathbb{R}$ which is continuous at a, $\lim_{\substack{x \to a \\ x \in G}} E_x[f(B(\tau_G))] = f(a)$

Proof.

Need: $\lim_{x \to a,\ x \in G} P_x\left(|B(\tau_G) - x| < \delta\right) = 1$

$$\begin{aligned} P_x(|B(\tau_G) - x| < \delta) &\geq P_x(\sup_{0 \le t \le \epsilon} |B(t) - x| < \delta,\ \tau_G \le \epsilon) \\ &\geq P_x(\sup_{0 \le t \le \epsilon} |B(t) - x| < \delta) - P_x(\tau_G \le \epsilon) \\ &\to 1 \ \text{ as } \ x \to a, x \in G \ \text{ then } \ \epsilon \downarrow 0 \end{aligned}$$

Proposition

$a \in \partial G$ is *regular* if $P_a(\sigma_G = 0) = 1 \quad \sigma_G = \inf\{t > 0 \ : \ B(t) \notin G\}$
a regular $\Leftrightarrow \quad \lim_{x \to a,\ x \in G} E_x[f(B_{\tau_G})] = f(a)\ \forall f$ bdd mble, cont at a

Proof of $\Rightarrow$

Enough to prove $P_x(\sigma_G < \epsilon)$ lower semi-continuous in x
Then $\limsup_{\substack{x \to a \\ x \in G}} P_x(\sigma_G < \epsilon) \geq P_a(\sigma_G < \epsilon) = 1$ and $\sigma_G \geq \tau_G$
But $\int p(0, x, \delta, y) P_y(\exists s \in (0, \epsilon - \delta), B(s) \notin G)$ continuous
and $\uparrow P_x(\sigma_G < \epsilon)$ as $\delta \downarrow 0$

Examples

1. If ∂G is a smooth manifold near a then a is regular by LIL
2. If $\exists$ cone C of height $h > 0$ and vertex at a such that $C - \{a\} \subset \bar{G}^C$ then a is a regular (exterior cone condition)
3. $d \leq 2$ always, $d \geq 3$ $\exists$counterexamples

Application to recurrence/transience of Brownian motion

$$G = \{y \in \mathbf{R}^d \ : \ \delta < |y| < R\} \qquad f = \begin{cases} 0 & |y| = R \\ 1 & |y| = \delta \end{cases}$$

$$u(x) = E_x[f(B(\tau_G))] = P_x(\tau_\delta < \tau_R) = \begin{cases} \frac{\log R - \log |x|}{\log R - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} & d > 2 \end{cases}$$

Theorem In $d \geq 2$, Brownian motion does not visit a point

Proof.

$$P_x(\tau_0 < \tau_R) = \lim_{\delta \downarrow 0} P_x(\tau_\delta < \tau_R) = \lim_{\delta \downarrow 0} \frac{\log R - \log |x|}{\log R - \log \delta} = 0$$

□

Theorem In $d = 2$, Brownian motion is recurrent, ie. comes arbitrarily close to any point arbitrarily many times

Proof.

$$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} P_x(\tau_\delta < \tau_R) = \lim_{R\uparrow\infty} \frac{\log R - \log|x|}{\log R - \log\delta} = 1$$

□

Theorem In $d \geq 3$, Brownian motion wanders off to infinity

Proof.

$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} = \left(\frac{|x|}{\delta}\right)^{2-d}$ if $|x| > \delta$,0 otherwise

$P_x(\text{hit } |y| = \delta \text{ after time } t) = \int \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} P_y(\tau_\delta < \infty) dy \to 0$ as $t \to \infty$

$P_x(\liminf_{t\to\infty} |B(t)| > \delta) = 1$

$\delta \uparrow \infty \liminf_{t\to\infty} |B(t)| = \infty \qquad a.s.$

□