Existence and Uniqueness Theorem

$\sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be Borel measurable, $\exists A < \infty$,

$$\|\sigma(x,t)\| + |b(x,t)| \le A(1+|x|) \qquad x \in \mathbb{R}^d,\ 0 \le t \le T$$

and *Lipschitz*;

$$\|\sigma(x,t) - \sigma(y,t)\| + |b(x,t) - b(y,t)| \le A|x-y|.$$

$x_0 \in \mathbb{R}^d$ indep of B_t, $E[|x_0|^2] < \infty$.
Then there exists a unique solution X_t on $[0, T]$ to

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

and $E[\int_0^T |X_t|^2 dt] < \infty$.

Uniqueness means that if X_t^1 and X_t^2 are two solutions then

$$P(X_t^1 = X_t^2,\ 0 \le t \le T) = 1$$

Proof of Uniqueness

Suppose X_t^1 and X_t^2 are solutions

$$\begin{aligned} X_t^1 - X_t^2 &= \int_0^t (b(X_s^1, s) - b(X_s^2, s))ds + \int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s \\ &\quad + x_0^1 - x_0^2 \end{aligned}$$

$$E[|X_t^1 - X_t^2|^2] \leq 4E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2]$$
$$+4E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s|^2] + 4E[|x_0^1 - x_0^2|^2]$$

$$E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2] \leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds$$

$$\begin{aligned} E[\int_0^t |\sigma(X_s^1, s) - \sigma(X_s^2, s))|^2 ds] &= E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))ds|^2] \\ &\leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds \end{aligned}$$

Call $\phi(t) = E[|X_t^1 - X_t^2|^2]$

$$\phi(t) \leq 8A^2 \int_0^t \phi(s)ds + 4\phi(0)$$

$$\Phi(t) = \int_0^t \phi(s)ds$$

$$(e^{-8A^2t}\Phi(t))' = (\Phi'(t) - 8A^2\Phi(t))e^{-8A^2t} \leq 4\phi(0)e^{-8A^2t}$$

$$e^{-8A^2t}\Phi(t) \leq 4\phi(0)$$

$$\phi(t) \leq 8A^2\Phi(t) + 4\phi(0) \leq 8e^{8A^2t}\phi(0)$$

$$E[|X_t^1 - X_t^2|^2] \leq 8e^{8A^2t}E[|x_0^1 - x_0^2|^2]$$

For each $0 \leq t \leq T$, $X_t^1 = X_t^2$ a.s. so $X_t^1 = X_t^2$ for all rational $t \in [0, T]$ a.s. By continuity this implies that $X_t^1 = X_t^2$ for all $t \in [0, T]$ a.s.

Proof of Existence

$$X_0(t) \equiv x_0$$

$$X_n(t) = x_0 + \int_0^t \sigma(s, X_{n-1}(s))dB(s) + \int_0^t b(s, X_{n-1}(s))ds$$

$$E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2]$$

Doob's inequality
$$\le 4E[\int_0^T \|\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s))\|^2 ds]$$

$$+TE[\int_0^T |b(s, X_{n-1}(s)) - b(s, X_{n-2}(s))|^2 ds]$$

$$\le C \int_0^T E[|X_{n-1}(s) - X_{n-2}(s)|^2]ds$$

$$\le CTE[\sup_{0\le t\le T} |X_{n-1}(t) - X_{n-2}(t)|^2]$$

Proof.

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le CTE[\sup_{0\le t\le T}|X_{n-1}(t)-X_{n-2}(t)|^2]$$

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le \frac{(CT)^n}{n!}$$

$$P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}) \le 2^{2n}E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2]$$

summable

$$\text{Borel}-\text{Cantelli} \quad \Rightarrow \qquad P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}\ i.o.) = 0.$$

Hence for almost every ω, $X_n(t) = X_0(t) + \sum_{j=0}^{n-1}(X_{j+1}(t) - X_j(t))$ converges uniformly on $[0, T]$ to a limit $X(t)$ which solves the required stochastic integral equation □

Lipschitz condition is *not* necessary

Theorem

Let $d = 1$ and

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq C|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\leq C|x-y|^{\alpha}, \quad \alpha \geq 1/2 \end{aligned}$$

Then there exists a solution of $dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t$ and it is unique

But you do need *some* regularity

$\sigma(x) = \mathrm{sgn}(x)$ and $dX = \sigma(B)dB$ Not a stochastic differential equation
But X is a Brownian motion $dB = \sigma(B)dX$ *is* a stochastic differential equation
But also $d(-B) = \sigma(-B)dX$ so no uniqueness

Markov property

X_t can be obtained by solving the stochastic differential equation up to time $s < t$ and then solving in $[s, t]$ with initial condition X_s
By uniqueness this gives the same answer
Define the transition probability

$$p(s, x, t, A) = P(X_t^{s,x} \in A)$$

where $X_t^{s,x}$ is the solution starting at x at time s
From the construction we have

$$P(X_t^{0,x} \in A \mid \mathcal{F}_s) = p(s, X_s^{0,x}, t, A)$$

which is the Markov property

Diffusions

A diffusion is a Markov process with transition probabilities $p(s, x, t, dy)$ satisfying, for each $\delta > 0$ as $h \to 0$,

$$i. \quad \frac{1}{h}\int_{|y-x|\geq\delta} p(t, x, t+h, dy) \to 0 \qquad \Rightarrow \text{ continuous paths}$$

$$ii. \quad \frac{1}{h}\int_{|y-x|<\delta} (y-x)p(t, x, t+h, dy) \to b(t, x)$$

$$iii. \quad \frac{1}{h}\int_{|y-x|<\delta} (y_i - x_i)(y_j - x_j)p(t, x, , t+h, dy) \to a_{ij}(t, x)$$

Formal derivation of the backward equation

$$p(s,x,t,A) = \int p(s,x,s+h,dy)p(s+h,y,t,A)$$

$$0 = \int p(s,x,s+h,dy)\Big\{p(s+h,y,t,A) - p(s,x,t,A)\Big\}$$

$$\begin{aligned}0 \;=\; & \int p(s,x,s+h,dy)\Big\{h\frac{\partial p(s,x,t,A)}{\partial s} + \sum_{i=1}^{d}(y_i - x_i)\frac{\partial p(s,x,t,A)}{\partial x_i} \\ & + \frac{1}{2}\sum_{i,j=1}^{d}(y_i - x_i)(y_j - x_j)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j} + \cdots\Big\}\end{aligned}$$

$$-\frac{\partial p(s,x,t,A)}{\partial s} = \sum_{i=1}^{d} b_i(t,x)\frac{\partial p(s,x,t,A)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j}$$

Real derivation

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \le s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time t

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,dz)$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$

$$u(s,x) = p(s,x,t,y) \quad \text{if} \quad -\frac{\partial}{\partial s}u = L_s u \quad 0 \le s < t \quad u(t,x) = \delta(x-y)$$

Existence result from PDE

Suppose that $a(t, x)$ and $b(t, x)$ are bounded and that there are $\alpha > 0$, $\gamma \in (0, 1]$, $C < \infty$ such that for all $s, t \geq 0$, $x, y \in \mathbb{R}^d$,

i. $\xi^T a(t, x)\xi \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d,$

ii. $\|a(s, x) - a(t, y)\| + |b(s, x) - b(t, y)| \leq C(|x - y|^\gamma + |t - s|^\gamma).$

Then the backward equation has a solution and furthermore

$$p(s, x, t, A) = \int_A p(s, x, t, y)dy$$

with $p(s, x, t, y) \geq 0$ jointly continuous in s, x, t, y. Furthermore, $p(s, x, t, y)$ is the unique weak solution of the forward equation,i.e.

$$\int f(t, y)p(s, x, t, y)dy - f(s, x) = \int_s^t \int \{\partial_u + \mathcal{L}\} f(u, y)p(s, x, u, y)dydu$$

The solution X_t, $t \geq 0$ of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with $X_0 = x$ is a Markov process with infinitesimal generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \qquad a = \sigma\sigma^*.$$

Itô's formula

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \Big\{\partial_s f(s, X_s) + \mathcal{L}f(s, X_s)\Big\} ds \\ &\quad + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s)\frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y)}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

Forward $$\frac{\partial p(s,x,t,y)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y)), \quad t > s,$$

$$p(s,x,s,y) = \delta(y-x)$$

Backward $$-\frac{\partial p(s,x,t,y)}{\partial s} = \frac{\sigma^2}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Under the previous conditions, the following are equivalent

1. $\exists B_t$, $t \geq 0$, $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$
2. For each $\lambda \in \mathbf{R}^d$,

$$Z_\lambda(t) = e^{\lambda\{X_t - \int_0^t b(s,X_s)ds\} - \frac{1}{2}\int_0^t \lambda^T a(s,X_s)\lambda ds}$$

   is a martingale with respect to $\mathcal{F}_t$
3. For all smooth $f(t,x)$,

$$f(t, X_t) - \int_0^t \{\partial_s + L\} f(s, X_s) ds$$

   is a martingale with respect to $\mathcal{F}_t$
4. For all smooth $f(x)$,

$$f(X_t) - \int_0^t Lf(X_s) ds$$

   is a martingale with respect to $\mathcal{F}_t$

$$dX_t = \sigma(X_t, t)dW_t + b(X_t, t)dt, \qquad X_0 = x$$

$$du(T-t, X_t) = \left\{ -\frac{\partial u}{\partial t} + Lu \right\} (T-t, X_t)dt + \sigma(X_t, t)\nabla u(X_t, t)dB_t.$$

$$\frac{\partial u}{\partial t} = \mathcal{L}u, \quad t > 0 \quad u(0, x) = f(x) \quad \Rightarrow \quad u(T-t, X_t) = \text{ martingale}$$

$$E_x[f(X_T)] = u(T, x)$$

Black-Scholes

Price at time t of European call option , maturity T , strike price K

$$V(t, S_t) = e^{-r(T-t)} E_{t, S_t}[(S_T - K)_+]$$

$$dS_t = rS_t dt + \sigma S_t dB_t \qquad \text{Geometric Brownian motion}$$

r = riskless interest rate , σ = stock volatility

$$V(t,S_t) = e^{-r(T-t)}E_{t,S_t}[(S_T - K)_+]$$

$$E_{t,x}[(S_t - K)_+] = \int (y-K)_+ p(T-t,x,y)dy$$

$$\frac{\partial V}{\partial t} = rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rV, \qquad V(T,S_T) = (S_T - K)_+$$

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

$$\begin{aligned} V(t,S_t) &= S_t \Phi\left(\frac{\log\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - e^{-r(T-t)} K \Phi\left(\frac{\log\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

Strong Markov Property

Let τ be a stopping time

$$f(X_t) - \int_0^t Lf(X_u)du = martingale$$

$$E[f(X_{t+\tau}) - \int_0^t Lf(X_{u+\tau})du \mid \mathcal{F}_{\tau+s}]$$
$$= f(X_{s+\tau}) - \int_0^s Lf(X_{u+\tau})du \qquad \text{optional stopping}$$

$\Rightarrow \tilde{X}_t = X_{\tau+t}$, $t \geq 0$ is a solution of

$$d\tilde{X}_t = b(t+\tau, \tilde{X}_t)dt + \sigma(t+\tau, \tilde{X}_t)d\tilde{B}_t \qquad t \geq 0$$

Generalized Dirichlet problem.

Let D be a domain in $\mathbb{R}^d$, i.e. a bounded connected open set with a smooth boundary ∂D
Suppose

$$\begin{cases} Lu = 0 & in\ D, \\ u = f & on\ \partial D \end{cases}$$

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial u}{\partial x_i}$$

Let X_t be the solution of the stochastic differential equation

$$dX_t = \sigma(X_t)dW_t + b(X_t)dt, \qquad X_0 = x.$$

Let τ be the exit time from the region D
Then X_τ is the exit point on the boundary ∂D

$$u(x) = E_x[f(X_\tau)]$$

To show it we cannot use the same conditioning argument as with Brownian motion because we don't have the symmetry anymore

Ito's formula:

$$u(X_{t\wedge\tau}) = \textit{ martingale}$$

Optional stopping

$$E_x[f(X_\tau)] = E_x[u(X_\tau)] = u(x)$$

Poisson equation

Suppose

$$\begin{cases} Lu = f & in\ D, \\ u = 0 & on\ \partial D \end{cases}$$

where f is some given function defined in D

Ito's formula:

$$u(X_{t\wedge\tau}) - \int_0^{t\wedge\tau} f(X_s)ds = \ martingale$$

Optional stopping: $E_x[u(X_\tau) - u(X_0) - \int_0^\tau f(X_s)ds] = 0.$

$$u(x) = E_x[\int_0^\tau f(X_s)ds]$$

Diffusions on manifolds a little tricky to define

Example. Brownian motion on the circle

If (X_1, X_2) is a point on the unit circle then the tangent at that point is $(-X_2, X_1)$. Therefore it would seem to make sense that the solution of

$$\begin{cases} dX_1 = -X_2 dB \\ dX_2 = X_1 dB \end{cases}$$

would be a Brownian motion on the circle. However we have

$$d(X_1^2(t) + X_2^2(t)) = 2(X_1^2(t) + X_2^2(t))dt$$

so it is not staying on the circle. The true Brownian motion on the circle,

$$(Y_1(t), Y_2(t)) = (\cos(B_t), \sin(B_t))$$

instead satisfies

$$\begin{cases} dY_1 = -\frac{1}{2} Y_1 dt - Y_2 dB \\ dY_2 = -\frac{1}{2} Y_2 dt + Y_1 dB \end{cases}$$

Martingale representation theorem

$\Omega = C[0, T]$, $\mathcal{F}_T$ = smallest σ-field with respect to which B_s are all measurable, $s \leq T$, P the Wiener measure , B_t = Brownian motion

M_t square integrable martingale with respect to $\mathcal{F}_t$

Then there exists $\sigma(t, \omega)$ which is

1. progressively measurable
2. square integrable
3. $\mathcal{B}([0, \infty)) \times \mathcal{F}$ mble

such that

$$M_t = M_0 + \int_0^t \sigma(s) dB_s$$