Feynman-Kac formula

$V$ a nice function (say bounded). $u \in C^{1,2}$ solves

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + Vu, \quad u(0, x) = u_0(x)$$

$$\int u_0(x) \exp\{-x^2/2t\} dx < \infty.$$ Then

$$u(t, x) = E_x[\exp\{\int_0^t V(B(s)) \, ds\} u_0(B(t))]$$

Proof.

For $0 \leq s \leq t$ let $Z(s) = u(t - s, B(s)) \exp\{\int_0^s V(B(u)) \, du\}$. By Itô’s lemma

$$Z(t) - Z(0) = \int_0^t \left\{ -\frac{\partial u}{\partial s} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + Vu \right\} \exp\{\int_0^s V(B(u)) \, du\} ds = 0$$

$$+ \int_0^t \frac{\partial u}{\partial x}(t - s, B(s)) \exp\{\int_0^s V(B(u)) \, du\} dB(s) = \text{mart}$$

$$\Rightarrow \quad E_x[Z(t)] = E_x[Z(0)]$$
In $d > 1$ if $u \in C^{1,2}$ solves
\[
\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u + Vu, \quad u(0, x) = u_0(x)
\]
then
\[
u(t, x) = E_x \left[ e^{\int_0^t V(B(s))ds} u_0(B(t)) \right]
\]
where $B(t)$ is $d$-dimensional Brownian motion.

If $V = V(t, x)$
\[
u(t, x) = E_x \left[ e^{\int_0^t V(t-s, B(s))ds} u_0(B(t)) \right]
\]

Feynman: solution $u$ of it Schrödinger equation $\frac{\partial u}{\partial t} = i [\frac{1}{2} \frac{\partial^2 u}{\partial x^2} + Vu]$ has "representation"
\[
u(x, t) = \int_{f: f(0) = x} e^{i \int_0^t V(f_s)ds - \frac{i}{2} \int_0^t |f'|^2 ds} u_0(f_t) d\mu
\]
where $\mu$ is translation invariant measure on space of functions.

BUT, No such measure $\mu$.

Kac pointed out that it is rigorous if $i \mapsto 1$ because
\[
e^{-\frac{1}{2} \int_0^t |f'|^2 ds} d\mu \text{ formally} \Rightarrow \text{Brownian motion}
\]
The Monte Carlo Method

Problem. Given $V$ compute

$$\lambda_0 = \sup \text{ spectrum}(\frac{1}{2}\Delta + V)$$

Idea (Ulam, Fermi, von Neumann, Metropolis)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Vu, \quad u(0, x) = 1$$

$$\lambda_0 = \lim_{t \to \infty} \frac{1}{t} \log u(0, t) = \lim_{t \to \infty} \frac{1}{t} \log E_0[e^{\int_0^t V(B(s))ds}]$$

Simulate $N$ Brownian paths $B_1, \ldots B_N$, $N$ large
Take $t$ large

$$\lambda_0 \approx \frac{1}{t} \log \frac{1}{N} \sum_{i=1}^N e^{\int_0^t V(B_i(s))ds}$$
Arcsin law

\[ \xi(t) = \frac{1}{t} \int_0^t 1_{[0,\infty)}(B(s)) \, ds = \text{the fraction of time that Brownian motion is positive up to time } t \]

\[ P(\xi(t) \leq a) = \begin{cases} 
0 & a < 0; \\
\frac{2}{\pi} \arcsin \sqrt{a} & 0 \leq a \leq 1; \\
1 & a > 1. 
\end{cases} \]

Simple explanation why distribution of \( \xi(t) \) indep of \( t \)

\[ \xi(t) = \int_0^1 1_{[0,\infty)}(B(t s)) \, ds = \int_0^1 1_{[0,\infty)}\left(\frac{1}{\sqrt{t}} B(t s)\right) \, ds = \int_0^1 1_{[0,\infty)}(\tilde{B}(s)) \, ds \]

\[ \xi(t) \overset{d}{=} \xi(1) \]
Proof

By Feynman-Kac if we can find a nice solution of

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} - u1_{[0, \infty)}
\]

\[u(0, x) = 1\]

Then

\[u(t, x) = E_x\left[e^{-\int_0^t 1_{[0, \infty)}(B(s))ds}\right]\]

and

\[u(t, 0) = \int_0^1 e^{-at} dP(\xi \leq a)\]

\[\alpha > 0, \phi_\alpha(x) = \alpha \int_0^\infty u(t, x) e^{-\alpha t} dt \quad \longrightarrow \quad -\frac{1}{2} \phi''_\alpha + (\alpha + 1_{[0, \infty)}) \phi_\alpha = \alpha\]

\[\phi_\alpha(x) = \begin{cases} 
\frac{\alpha}{\alpha+1} + A e^{x\sqrt{2(\alpha+1)}} + B e^{-x\sqrt{2(\alpha+1)}} & , x \geq 0, \\
1 + C e^{x\sqrt{2\alpha}} + D e^{-x\sqrt{2\alpha}} & , x \leq 0.
\end{cases}\]

\[u \leq 1 \Rightarrow \phi_\alpha \leq 1 \Rightarrow A = D = 0\]
Proof.

\[ \phi_\alpha(0^-) = \phi_\alpha(0^+), \phi'_\alpha(0^-) = \phi'_\alpha(0^+) \]

\[ \Rightarrow B = \frac{\alpha^{1/2}}{(1+\alpha)(\sqrt{\alpha}+\sqrt{\alpha+1})}, \quad C = \frac{1}{(1+\alpha)^{1/2}(\sqrt{\alpha}+\sqrt{\alpha+1})^{1/2}} \]

\[ \phi_\alpha(0) = \sqrt{\frac{\alpha}{\alpha + 1}} = \int_0^\infty E[e^{-t\xi}\alpha e^{-\alpha t}]dt \]

By Fubini’s theorem this reads \( E\left[\frac{\alpha}{\alpha + \xi}\right] = \sqrt{\frac{\alpha}{\alpha + 1}} \) or

\[ \int_0^1 \frac{1}{1 + \gamma a} dP(\xi \leq a) = \frac{1}{\sqrt{1 + \gamma}} \]

Looking up a table of transforms we find

\[ dP(\xi \leq a) = \frac{2}{\pi} \frac{1}{\sqrt{a(1 - a)}} da \quad 0 \leq a \leq 1 \]

which is the density of the arcsin distribution

\[ \square \]
Stochastic differential equations

\[ \sigma(x, t), \ b(x, t) \text{ mble} \]

**Definition**

A stochastic process \( X_t \) is a solution of a stochastic differential equation

\[
dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \quad X_0 = x_0
\]

on \([0, T]\) if \( X_t \) is progressively measurable with respect to \( \mathcal{F}_t \),

\[
\int_0^T |b(X_t, t)|dt < \infty, \ \int_0^T |\sigma(X_t, t)|^2dt < \infty \text{ a.s. and }
\]

\[
X_t = x_0 + \int_0^t b(X_s, s)ds + \int_0^T \sigma(X_s, s)dB_s \quad 0 \leq t \leq T
\]

The main point is that \( \sigma(\omega, t) = \sigma(X_t, t), \ b(\omega, t) = b(X_t, t) \)

Under reasonable conditions the solution \( X_t \) exists, is unique, and is a Markov process.
Ornstein-Uhlenbeck Process

\( X_t, t \geq 0 \) is the solution of the *Langevin equation*

\[
dX_t = -\alpha X_t dt + \sigma dB_t, \quad \alpha > 0
\]

To solve it

\[
de^{\alpha t} X_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dB_t
\]

so

\[
X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha (t-s)} dB_s
\]

If \( X_0 \sim \mathcal{N}(m, V) \) indep of \( B_t, t \geq 0 \) \( \Rightarrow \) \( X_t \) Gaussian process

\[
m(t) = E[X_t] = me^{-\alpha t}
\]

\[
c(s, t) = \text{Cov}(X_s, X_t) = [V + \frac{\sigma^2}{2\alpha}(e^{2\alpha \min(t,s)} - 1)]e^{-\alpha (t+s)}
\]

\( m = 0, V = \frac{\sigma^2}{2\alpha} \Rightarrow X_t \) stationary Gaussian

\[
c(s, t) = \frac{\sigma^2}{2\alpha} e^{-\alpha (t-s)}
\]

\[
Y_t = \int_0^t X_s ds \quad \text{"Physical" Brownian motion}
\]
Geometric Brownian motion

\[ dS_t = \mu S_t dt + \sigma S_t dB_t \quad \mu = \text{drift} \quad \sigma = \text{volatility} \]

By Ito’s formula

\[ S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t} \]

is the solution

\( S_t \geq 0 \) so it is (Samuelson) a better model of stock prices than \( B_t \) (Bachelier)

Sometimes people write

\[ \frac{dS_t}{S_t} = \mu dt + \sigma dB_t \]

but note that \( \frac{dS_t}{S_t} \neq d \log S_t \)
Bessel process \((d = 2)\)

Let \(B_t = (B^1_t, B^2_t)\) be 2d Brownian motion starting at 0,

\[ r_t = |B_t| = \sqrt{(B^1_t)^2 + (B^2_t)^2}. \]

By Ito’s lemma,

\[ dr_t = \frac{B^1_t}{|B_t|} dB^1_t + \frac{B^2_t}{|B_t|} dB^2_t + \frac{1}{2} \frac{1}{|B_t|} dt. \]

This is \textit{not} a stochastic differential equation.
\[ Y(t) = \int_0^t \frac{B^1}{|B|} dB^1 + \int_0^t \frac{B^2}{|B|} dB^2 \]

Let \( f(t, y) \) be a smooth function. Use Itô's lemma. Intuitively

\[
df(t, Y_t) = \partial_t f dt + \partial_y f dY + \frac{1}{2} \partial_y^2 f (dY)^2
\]

\[
(dY)^2 = \left( \frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2 \right)^2
\]
\[
= \left( \frac{B^1}{|B|} \right)^2 (dB^1)^2 + 2 \frac{B^1 B^2}{|B|^2} dB^1 dB^2 + \left( \frac{B^2}{|B|} \right)^2 (dB^2)^2
\]
\[
= dt
\]

\[
f(t, Y_t) = f(0, Y_0) + \int_0^t \left( \partial_t f + \frac{1}{2} \partial_y^2 f \right)(s, Y_s) ds
\]
\[
\quad + \int_0^t \partial_y f \frac{B^1}{|B|} dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|} dB^2
\]
Itô’s lemma

\[ dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt \]

\[ f(t, X_t) = f(0, X_0) + \int_0^t \left\{ \partial_s f(s, X_s) + \mathcal{L} f(s, X_s) \right\} ds \]

\[ + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \]

\[ \mathcal{L} f(t, x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^{d} b_i(t, x) \frac{\partial f}{\partial x_i}(t, x) \]

\[ a_{ij} = \sum_{k=1}^{d} \sigma_{ik} \sigma_{jk} \quad a = \sigma \sigma^T \]
\[ dr_t = \frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2 + \frac{1}{2} \frac{1}{|B|} dt = dY_t + \frac{1}{2} r_t^{-1} dt \]

\[ f(t, Y_t) = f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2} \partial_y^2 f)(s, Y_s) ds \]
\[ + \int_0^t \partial_y f \frac{B^1}{|B|} dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|} dB^2 \]

In particular \( e^{\lambda Y_t - \lambda^2 t/2} \) is a martingale

So \( Y_t \) is a Brownian motion.

Therefore
\[ dr_t = dY_t + \frac{1}{2} r_t^{-1} dt \]

is a stochastic differential equation for the new Brownian motion \( Y_t \)
Itô’s lemma

\[ f(t, X_t) - f(0, X_0) = \int_0^t \left\{ \partial_s + \mathcal{L} \right\} f(s, X_s) \, ds + \int_0^t \nabla f(s, X_s) \cdot \sigma \, dB_s \]

Proof

\[ = \sum_i f(t_{i+1}, X_{t_{i+1}}) - f(t_i, X_{t_i}) \]

\[ = \sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) + \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i}) \]

\[ + \frac{1}{2} \sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X_{t_{i+1}}^j - X_{t_i}^j)(X_{t_{i+1}}^k - X_{t_i}^k) \]

+ higher order terms
Proof continued

\[
\sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) \to \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds
\]

\[
\sum_i \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})
\]

\[
= \sum_i \nabla f(t_i, X_{t_i}) \cdot \left( \int_{t_i}^{t_{i+1}} \sigma(s, X_s)dB_s \right) \to \int_0^t \nabla f \cdot \sigma dB
\]

\[
+ \sum_i \nabla f(t_i, X_{t_i}) \cdot \left( \int_{t_i}^{t_{i+1}} b(s, X_s)ds \right) \to \int_0^t \nabla f \cdot b ds
\]

\[
\sum_i \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X_{t_{i+1}}^j - X_{t_i}^j)(X_{t_{i+1}}^k - X_{t_i}^k) \to \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(s, X_s)a_{jk}(s, X_s)ds
\]
Proof continued

To show the last convergence, ie

\[
\sum_i g(t_i, X_{t_i})(X_{t_{i+1}}^j - X_{t_i}^j)(X_{t_{i+1}}^k - X_{t_i}^k) \to \int_0^t g(s, X_s)a_{jk}(s, X_s)ds
\]

\[
Z(t_i, t_{i+1}) = \left( \int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}(s, X_s)dB_s^l \right) \left( \int_{t_i}^{t_{i+1}} \sum_m \sigma_{km}(s, X_s)dB_s^m \right) - \int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}\sigma_{kl}(s, X_s)ds
\]

\[
E[|Z(t_i, t_{i+1})|^2] = \mathcal{O}((t_{i+1} - t_i)^2)
\]
Proof continued

\[ \sum_i g(t_i, X_{t_i}) E[\int_{t_i}^{t_{i+1}} a_{ij}(s, X_s) ds] \rightarrow \int_0^t g(s, X_s) a_{ij}(s, X_s) ds \]

\[ E[(\sum_i g(t_i, X_{t_i}) Z(t_i, t_{i+1}))^2] = \sum_{i,j} E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1})] \]

\[ i < j \quad E[E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1}) \mid \mathcal{F}_{t_j}]] = 0 \]

\[ i = j \quad E[E[g^2(t_i, X_{t_i}) Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \]

\[ = E[g^2(t_i, X_{t_i}) E[Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \]

\[ = O((t_{i+1} - t_i)^2) \]
\[
f(t, X_t) - f(0, X_0) - \int_0^t \{ \partial_s + \mathcal{L} \} f(s, X_s) ds = \int_0^t \nabla f(s, X_s) \cdot \sigma dB_s
\]

\[
\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(t, x) \frac{\partial}{\partial x_i} = \text{generator}
\]

\[
M_t = f(t, X_t) - \int_0^t \{ \partial_s + \mathcal{L} \} f(s, X_s) ds \text{ is a martingale}
0 = E[f(t, X_t) - f(s, X_s) - \int_s^t \{ \partial_u + \mathcal{L} \} f(u, X_u) du | \mathcal{F}_s]
\]

\[
= \int f(t, y)p(s, x, t, y) dy - f(s, x)
\]

\[
- \int_s^t \int \{ \partial_u + \mathcal{L} \} f(u, y)p(s, x, u, y) dydu, \quad X_s = x
\]
For any $f$,

$$0 = \int f(t, y)p(s, x, t, y)dy - f(s, x)$$

$$- \int_s^t \int \{ \partial u + L \} f(u, y)p(s, x, u, y)dydu$$

**Fokker-Planck (Forward) Equation**

$$\frac{\partial}{\partial t}p(s, x, t, y) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial y_i \partial y_j}(a_{i,j}(t, y)p(s, x, t, y))$$

$$- \sum_{i=1}^d \frac{\partial}{\partial y_i}(b_i(t, y)p(s, x, t, y))$$

$$= L^*_y p(s, x, t, y)$$

$$\lim_{t \downarrow s} p(s, x, t, y) = \delta(y - x).$$
Kolmogorov (Backward) Equation

\[-\frac{\partial}{\partial s} p(s, x, t, y) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(s, x) \frac{\partial^2 p(s, x, t, y)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s, x) \frac{\partial p(s, x, t, y)}{\partial x_i} = L_x p(s, x, t, y)\]

\[\lim_{s \uparrow t} p(s, x, t, y) = \delta(y - x).\]
Proof.

$f(x)$ smooth

$$-\frac{\partial}{\partial s} u = L_s u \quad 0 \leq s < t \quad u(t, x) = f(x)$$

Ito’s formula: $u(s, X(s))$ martingale up to time $t$

$$u(s, x) = E_{s,x}[u(s, X(s))] = E_{s,x}[u(t, X(t))] = \int f(z) p(s, x, t, z) dz$$

Let $f_n(z)$ smooth functions tending to $\delta(y - z)$. We get in the limit that $p$ satisfy the backward equations.
Example. Brownian motion $d = 1$

\[ \mathcal{L} = \frac{1}{2} \frac{\partial^2}{\partial x^2} \]

**Forward** \( \frac{\partial p(s, x, t, y)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial y^2}, \quad t > s \)

\[ p(s, x, s, y) = \delta(y - x) \]

**Backward** \( -\frac{\partial p(s, x, t, y)}{\partial s} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial x^2}, \quad s < t, \)

\[ p(t, x, t, y) = \delta(y - x) \]
Example. Ornstein-Uhlenbeck Process

\[
\mathcal{L} = \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}
\]

**Forward**
\[
\frac{\partial p(s, x, t, y)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial y^2} + \frac{\partial}{\partial y} (\alpha y p(s, x, t, y)), \quad t > s,
\]
\[
p(s, x, s, y) = \delta(y - x)
\]

**Backward**
\[
-\frac{\partial p(s, x, t, y)}{\partial s} = \frac{1}{2} \frac{\partial^2 p(s, x, t, y)}{\partial x^2} - \alpha x \frac{\partial p(s, x, t, y)}{\partial x}, \quad s < t,
\]
\[
p(t, x, t, y) = \delta(y - x)
\]