Definition: Progressively measurable

$\sigma(s,\omega)$ is called *progressively measurable* if

1. i. $\sigma(s,\omega)$ is $\mathcal{B}[0,\infty) \times \mathcal{F}$ measurable;
2. ii. For all $t \geq 0$, the map $[0,t] \times \Omega \to \mathbb{R}$ given by $\sigma(s,\omega)$ is $\mathcal{B}[0,t] \times \mathcal{F}_t$ measurable.

$\mathcal{B}[0,t]$ denotes the Borel σ-algebra on $[0,t]$.
Informally, $\sigma(s,\omega)$ is *nonanticipating*= uses information about ω contained in $\mathcal{F}_s$.

Definition: Simple Functions

$\sigma(s,\omega)$ is called *simple* if there exists a partition $0 \leq s_0 < s_1 < \cdots$ of $[0,\infty)$ and bounded random variables $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ such that $\sigma(s,\omega) = \sigma_j(\omega)$ for $s_j \leq s < s_{j+1}$.

Definition: Stochastic Integral for Simple Functions

Given such a $\sigma(s,\omega) = \sigma_j(\omega)$ for $s_j \le s < s_{j+1}$, $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ define

$$\int_0^t \sigma(s,\omega)dB(s) = \sum_{j=0}^{J(t)-1} \sigma_j(\omega)(B(s_{j+1})-B(s_j)) + \sigma_{J(t)}(\omega)(B(t)-B(s_{J(t)}))$$

where $s_{J(t)} < t \le s_{J(t)+1}$.

Basic properties

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale

Proof.

Since $\sigma_j \in \mathcal{F}_{s_j}$, if $u \ge s_j$, $E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_u] = \sigma_j(B(u) - B(s_j))$ and if $u < s_j$, $E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_u] =$ $E[E[\sigma_j(B(s_{j+1}) - B(s_j)) \mid \mathcal{F}_{s_j}] \mid \mathcal{F}_u] = 0$. □

Basic properties

3 $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$

Proof.

$\int_0^t \sigma dB = \sum_j \sigma_j (B(s_{j+1} \wedge t) - B(s_j))$

$E[(\int_0^t \sigma dB)^2] = \sum_{i,j} E[\sigma_i \sigma_j (B(s_{i+1} \wedge t) - B(s_i))(B(s_{j+1} \wedge t) - B(s_j))]$

$i < j : E[E[\sigma_i \sigma_j (B(s_{i+1} \wedge t) - B(s_i))(B(s_{j+1} \wedge t) - B(s_j)) \mid \mathcal{F}_{s_j}]] = 0$

$i = j : E[\sigma_j^2 (B(s_{j+1} \wedge t) - B(s_j))^2] =$
$E[E[\sigma_j^2 (B(s_{j+1} \wedge t) - B(s_j))^2 \mid \mathcal{F}_{s_j}] = E[\sigma_j^2](s_{j+1} \wedge t - s_j)$

$E[(\int_0^t \sigma dB)^2] = \sum_j E[\sigma_j^2 (B(s_{j+1} \wedge t) - B(s_j))^2] = E[\int_0^t \sigma^2 ds]$

□

Basic properties

4 $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale

Proof.

Suppose $t \geq u \geq J(t)$. Then $E[e\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds \mid \mathcal{F}_u]$ can be written

$$e^{\sum_{j=0}^{J(t)-1} \sigma_j(B(s_{j+1})-B(s_j))-\frac{1}{2}\sigma_j^2(s_{j+1}-s_j)} E[e^{\sigma_{J(t)}(B(t)-B(s_{J(t)})-\frac{1}{2}\sigma_{J(t)}^2(t-s_{J(t)})} \mid \mathcal{F}_u].$$

The expectation is just 1 , so we have that $E[Z(t) \mid \mathcal{F}_u] = Z(u)$ whenever $t \geq u \geq J(t)$. It follows by repeated conditioning that $E[Z(t) \mid \mathcal{F}_u] = Z(u)$ for any $u \leq t$. □

$\mathcal{P}$ = set of progressively measurable functions

Lemma

For each $t.0$, $\mathcal{P} =$ Closure in $L^2([0,t] \times \Omega, dt \times dP)$ of simple functions

Proof

Suppose $\sigma \in \mathcal{P}$ and $E[\int_0^t \sigma^2(s,\omega)ds] < \infty$
we need to find a sequence σ_n of simple functions s.t.

$$E[\int_0^t (\sigma(s,\omega) - \sigma_n(s,\omega))^2 ds] \to 0.$$

We can assume that σ is bounded For if $\sigma_N = \sigma$ for $|\sigma| \leq N$ and 0 otherwise then $\sigma_N \to \sigma$ and $|\sigma_N - \sigma|^2 \leq 4|\sigma|^2$ so by the dominated convergence theorem $E[\int_0^t (\sigma - \sigma_N)^2 ds] \to 0$.

Proof.

Furthermore we can assume that σ is continuous in s

for if σ is bounded then $\sigma_h = h^{-1} \int_{t-h}^{t} \sigma ds$ are continuous progressively measurable and converge to σ as $h \to 0$. By the bounded convergence theorem

$$E[\int_0^t (\sigma - \sigma_h)^2 ds] \to 0$$

For σ continuous bounded and progressively measurable let

$$\sigma_n(s, \omega) = \sigma(\frac{\lfloor ns \rfloor}{n}, \omega)$$

These are progressively measurable, bounded and simple functions converging to σ and again by the bounded convergence theorem,

$$E[\int_0^t (\sigma - \sigma_n)^2 ds] \to 0$$

Theorem (Definition of the Itô Integral)

Let $\sigma(s, \omega)$ be progressively measurable and for each $t \geq 0$, $E[\int_0^t \sigma^2 ds] < \infty$. Let σ_n be simple functions with $E[\int_0^t (\sigma_n - \sigma)^2 ds] \to 0$ and set

$$X_n(t, \omega) = \int_0^t \sigma_n(s, \omega) dB(s).$$

Then

$$X(t, \omega) = \lim_{n \to \infty} X_n(t, \omega)$$

exists uniformly in probability, i.e. for each $T > 0$ and $\epsilon > 0$,

$$\lim_{n \to \infty} P(\sup_{0 \leq t \leq T} |X_N(t, \omega) - X(t, \omega)| \geq \epsilon) = 0.$$

Furthermore the limit is independent of the choice of approximating sequence $\sigma_n \to \sigma$. The limit $X(t, \omega)$ is the Itô integral

$$X(t) = \int_0^t \sigma(s) dB(s)$$

Proof.

$X_n(t) - X_m(t) = \int_0^t (\sigma_n - \sigma_m) dB$ is a continuous martingale so by Doob's inequality

$$
\begin{aligned}
P(\sup_{0\le t\le T} |X_n(t) - X_m(t)| \ge \epsilon) &\le \epsilon^{-2} E[(X_n - X_m)^2(T)] \\
&= \epsilon^{-2} E[\int_0^T (\sigma_n - \sigma_m)^2 ds]
\end{aligned}
$$

So $X_n - X_m$ is uniformly Cauchy in probability and therefore there exists a progressively measurable X with

$$
P(\sup_{0\le t\le T} |X(t,\omega) - X_n(t,\omega)| \ge \epsilon) \stackrel{n\to\infty}{\to} 0 \qquad \epsilon > 0
$$

If $\sigma_n' \stackrel{L^2}{\to} \sigma$ and $X_n' = \int_0^t \sigma_n' dB$, $P(\sup_{0\le t\le T} |X_n - X_n'| \ge \epsilon) \to 0$ so that X_n and X_n' have the same limit. □

Basic properties of the Itô Integral

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale.
3. $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$.
4. If $|\sigma| \leq C$ then $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale

proof

1. By construction
2. Continuity follows from the construction. To prove the limit is a martingale we have $E[X_n(t) \mid \mathcal{F}_s] = X_n(s)$ and $X_n \to X$ in L^2, therefore in L^1 as well. The L^1 limit of a martingale is a martingale.
3. $X_n^2(t) - \int_0^t \sigma_n^2(s)ds$ is a martingale $\xrightarrow{L^1} X^2(t) - \int_0^t \sigma^2(s)ds$
4. $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$ is a martingale so it suffices to show that $Z_n(t)$, $n = 1, 2, \ldots$ is a uniformly integrable family.

Proof.

to show that $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$, $n = 1, 2, \ldots$ is a uniformly integrable family, it is enough to show that there is some fixed $C < \infty$ for which $E[(Z_N(t))^2] \leq C$.

$$\begin{aligned} E[(Z_N(t))^2] &= E[\exp\{2\int_0^t \sigma_n dB - \int_0^t \sigma_n^2 ds\}] \\ &\leq e^{Ct} E[\exp\{2\int_0^t \sigma_n dB - \frac{4}{2}\int_0^t \sigma_n^2 ds\}] \\ &= e^{Ct} \end{aligned}$$

□

A *stochastic integral* is an expression of the form

$$X(t,\omega) = \int_0^t \sigma(s,\omega)dB(s) + \int_0^t b(s,\omega)ds + X_0$$

where σ and b are progressively measurable with $E[\int_0^t \sigma^2(s,\omega)ds] < \infty$ and $\int_0^t |b(s,\omega)|ds < \infty$ for all $t \geq 0$, and $X_0 \in \mathcal{F}_0$ is the starting point

The stochastic differential

$$dX = \sigma dB + bdt$$

is shorthand for the same thing

For example the integral formula $\int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - t)$ can be written in differential notation as

$$dB^2 = 2BdB + dt$$

What happens if $B^2(t)$ is replaced by a more general function $f(B(t))$?

Itô's Lemma

Let $f(x)$ be twice continuously differentiable. Then

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt$$

Proof

First of all we can assume without loss of generality that f, f' and f'' are all uniformly bounded , for if we can establish the lemma in the uniformly bounded case, we can approximate f by f_n so that all the corresponding derivatives are bounded and converge to those of f uniformly on compact sets.

Let $s = t_0 < t_1 < t_2 < \cdots < t_n = t$. We have

$$\begin{aligned} f(B(t)) - f(B(s)) &= \sum_{j=0}^{n-1} [f(B(t_{j+1})) - f(B(t_j))] \\ &= \sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \\ \xi_j \in [t_j, t_{j+1}] \qquad & \quad + \sum_{j=0}^{n-1} \frac{1}{2} f''(B(\xi_j))(B(t_{j+1}) - B(t_j))^2, \end{aligned}$$

Let the width of the partition go to zero. By definition of the stochastic integral

$$\sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_s^t f' dB.$$

As in the computation of the quadratic variation,

$$E\left[\left(\sum_{j=0}^{n-1} f''(B(\xi_j))\left[(B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right]\right)^2\right]$$
$$=\sum_{j=0}^{n-1} E\left[\left(f''(B(\xi_j))\right)^2\left[(B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right]^2\right]+o(1)\to 0$$

Hence

$$\sum_{j=0}^{n-1} f''(B(t_j))(B(t_{j+1})-B(t_j))^2 \xrightarrow{L^2} \int_s^t f''(B(u))du$$

So we have proved that

$$f(B(t)-f(B(s))=\int_s^t f'(B(u))dB(u)+\frac{1}{2}\int_s^t f''(B(u))du$$

which is Itô's formula.

1. In differential notation Itô's formula reads

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt.$$

The Taylor series is $df(B) = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(B)(dB)^n$. In normal calculus we would have $(dB)^n = 0$ if $n \geq 2$, but because of the finite quadratic variation of Brownian paths we have $(dB)^2 = dt$, while still $(dB)^n = 0$ if $n \geq 3$.

2. If the function f depends on t as well as $B(t)$, the formula is

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \frac{\partial f}{\partial x}(t, B(t))dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B(t))dt.$$

The proof is about the same as the special case above.

3. If $B(t)$ is a d-dimensional Brownian motion and $f(t, x)$ is a function on $[0, \infty) \times \mathbf{R}^d$ which has one continuous derivative in t and two continuous derivatives in x, then the formula reads

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \nabla f((t, B(t)) \cdot dB(t) + \frac{1}{2}\Delta f(t, B(t))dt.$$

Local time

f continuous function on $\mathbb{R}_+$

$$L_t(x) = \int_0^t \delta_x(f(s))ds = \lim_{\epsilon \downarrow 0}(2\epsilon)^{-1}|\{0 \le s \le t : |f(s) - x| \le \epsilon\}|$$

$$\int_0^t 1_A(f(s))ds = \int_A L_t(x)dx$$

$f \in C^1$ $L_t(x) = \sum_{s_i \in [0,t]: f(s_i)=x} |f'(s_i)|^{-1}$ discontinuous in t
Itô 's lemma applied to $|B_t - x|$ gives

Tanaka's formula for Brownian Local Time

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \operatorname{sgn}(B_s - x)dB_s$$

In particular, $L_t(x)$ continuous in t a.s.

But $|x|$ not differentiable, so no fair!!!

Proof.

$$f''_\epsilon(x) = (2\epsilon)^{-1} 1_{[-\epsilon,\epsilon]}$$

Itô

$$(2\epsilon)^{-1}|\{0 \le s \le t : |B_s| \le \epsilon\}| = f_\epsilon(B_t) - f_\epsilon(B_0) - \int_0^t f'_\epsilon(B_s)dB_s$$

$$\epsilon \downarrow 0$$

$$L_t(x) = |B_t - x| - |B_0 - x| - \int_0^t \mathrm{sgn}(B_s - x)dB_s$$

Note: To be honest, we have to do a little bit more convolution to make f'' continuous. □

Feynman-Kac formula

V a nice function (say bounded). $u \in C^{1,2}$ solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu, \qquad u(0,x) = u_0(x)$$

$\int u(x) \exp\{-x^2/2t\} dx < \infty$. Then

$$u(t,x) = E_x\left[e^{\int_0^t V(B(s))ds} u_0(B(t))\right]$$

Proof.

For $0 \le s \le t$ let $Z(s) = u(t-s, B(s))e^{\int_0^s V(B(u))du}$. By Îto's lemma

$$\begin{aligned} Z(t) - Z(0) &= \int_0^t \left\{-\frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu\right\} e^{\int_0^s V(B(u))du} ds \qquad = 0 \\ &\quad + \int_0^t \frac{\partial u}{\partial x}(t-s, B(s))e^{\int_0^s V(B(u))du} ds \qquad = \text{martingale} \end{aligned}$$

so $E_x[Z(t)] = E_x[Z(0)]$ □

1. In $d > 1$ if $u \in C^{1,2}$ solves
$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Vu, \quad u(0,x) = u_0(x)$$
then
$$u(t,x) = E_x\left[e^{\int_0^t V(B(s))ds} u_0(B(t))\right]$$
where $B(t)$ is d-dimensional Brownian motion
2. If $V = V(t,x)$
$$u(t,x) = E_x\left[e^{\int_0^t V(t-s,B(s))ds} u_0(B(t))\right]$$
3. Historical remark. Feynman's thesis was that solution u of it Schrödinger equation $\frac{\partial u}{\partial t} = i[\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu]$ should have representation
$$u(x,t) = " \int " e^{i\int_0^t V(f_s)ds - \frac{i}{2}\int_0^t |f'|^2 ds} u_0(f_t)$$
where "$\int$" is supposed to be average over functions starting at x. Kac pointed out that it is rigorous if $i \mapsto 1$

Arcsin law

$\xi(t) = \frac{1}{t}\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds =$ the fraction of time that Brownian motion is positive up to time t

$$P(\xi(t) \le a) = \begin{cases} 0 & a < 0; \\ \frac{2}{\pi} \arcsin \sqrt{a} & 0 \le a \le 1; \\ 1 & a > 1. \end{cases}$$

Simple explanation why distribution of $\xi(t)$ indep of t

$$\xi(t) = \int_0^1 \mathbf{1}_{[0,\infty)}(B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\frac{1}{\sqrt{t}}B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\tilde{B}(s))ds$$

$$\xi(t) \stackrel{d}{=} \xi(1)$$

Proof

By Feynman-Kac if we can find a nice solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u\mathbf{1}_{[0,\infty)} \qquad u(0,x) = 1$$

Then

$$u(t,x) = E_x[e^{-\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds}]$$

and

$$u(t,0) = \int_0^1 e^{-at} dP(\xi \le a)$$

$\alpha > 0$, $\phi_\alpha(x) = \alpha \int_0^\infty u(t,x) e^{-\alpha t} dt \longrightarrow -\frac{1}{2}\phi_\alpha'' + (\alpha + \mathbf{1}_{[0,\infty)})\phi_\alpha = \alpha$

$$\phi_\alpha(x) = \begin{cases} \frac{\alpha}{\alpha+1} + Ae^{x\sqrt{2(\alpha+1)}} + Be^{-x\sqrt{2(\alpha+1)}}, & x \ge 0, \\ 1 + Ce^{x\sqrt{2\alpha}} + De^{-x\sqrt{2\alpha}}, & x \ge 0. \end{cases}$$

$u \le 1 \Rightarrow \phi_\alpha \le 1 \Rightarrow A = D = 0$

Proof.

$\phi_\alpha(0_-) = \phi_\alpha(0_+), \phi'_\alpha(0_-) = \phi'_\alpha(0_+)$

$\Rightarrow B = \frac{\alpha^{1/2}}{(1+\alpha)(\sqrt{\alpha}+\sqrt{\alpha+1})}, C = \frac{1}{(1+\alpha)^{1/2}(\sqrt{\alpha}+\sqrt{\alpha+1})^{1/2}}$

$$\phi_\alpha(0) = \sqrt{\frac{\alpha}{\alpha+1}} = \int_0^\infty E[e^{-t\xi}\alpha e^{-\alpha t}]dt$$

By Fubini's theorem this reads $E\left[\frac{\alpha}{\alpha+\xi}\right] = \sqrt{\frac{\alpha}{\alpha+1}}$or

$$\int_0^1 \frac{1}{1+\gamma a} dP(\xi \leq a) = \frac{1}{\sqrt{1+\gamma}}$$

Looking up a table of transforms we find

$$dP(\xi \leq a) = \frac{2}{\pi} \frac{1}{\sqrt{a(1-a)}} da \qquad 0 \leq a \leq 1$$

which is the density of the arcsin distribution □

Stochastic differential equations

$\sigma(x,t)$, $b(x,t)$ mble

Definition

A stochastic process X_t is a solution of a stochastic differential equation

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

on $[0,T]$ if X_t is progressively measurable with respect to $\mathcal{F}_t$, $\int_0^T |b(X_t,t)|dt < \infty$, $\int_0^T |\sigma(X_t,t)|^2 dt < \infty$ a.s. and

$$X_t = x_0 + \int_0^t b(X_s,s)ds + \int_0^T \sigma(X_s,s)dB_s \qquad 0 \le t \le T$$

The main point is that $\sigma(\omega,t) = \sigma(X_t,t)$, $b(\omega,t) = b(X_t,t)$

Ornstein-Uhlenbeck Process

X_t, $t \geq 0$ is the solution of the *Langevin equation*

$$dX_t = -\alpha X_t dt + \sigma dB_t$$

To solve it

$$de^{\alpha t} X_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dB_t$$

so

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

If $X_0 \sim \mathcal{N}(m, V)$ indep of B_t, $t \geq 0 \Rightarrow X_t$ Gaussian process

$$m(t) = E[X_t] = me^{-\alpha t}$$

$$c(s,t) = \mathrm{Cov}(X_s, X_t) = [V + \frac{\sigma^2}{2\alpha}(e^{2\alpha \min(t,s)} - 1)]e^{-\alpha(t+s)}$$

$m = 0,\ V = \frac{\sigma^2}{2\alpha} \Rightarrow X_t$ stationary Gaussian $c(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha(t-s)}$

$Y_t = \int_0^t X_s ds$ "Physical" Brownian motion

Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t \qquad \mu = \text{drift} \quad \sigma = \text{volatility}$$

By Ito's formula

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

Bessel process ($d = 2$)

Let $B_t = (B_t^1, B_t^2)$ be 2d Brownian motion starting at 0,

$$r_t = |B_t| = \sqrt{(B_t^1)^2 + (B_t^2)^2}.$$

By Ito's lemma,

$$dr_t = \frac{B^1}{|B|}dB^1 + \frac{B^2}{|B|}dB^2 + \frac{1}{2}\frac{1}{|B|}dt.$$

As it stands this is *not* a stochastic differential equation.
Let Y_t be the solution of

$$dY = \frac{B^1}{|B|}dB^1 + \frac{B^2}{|B|}dB^2, \qquad Y_0 = 0.$$

Let $f(x)$ be a smooth function and use Ito's lemma to show that

$$f(t, Y_t) - f(0, Y_0) - \frac{1}{2}\int_0^t (\partial_t f + \partial_x^2 f)(s, Y_s)ds$$

is a martingale