Brownian motion in $\mathbb{R}^d$

1. $B_t = (B_t^1, \ldots, B_t^d)$, B_t^i independent Brownian motions
2. B_t Markov with $P(B_t \in A \mid B_s = x) = \int_A \frac{1}{(2\pi(t-s))^{d/2}} e^{-\frac{|y-x|^2}{2(t-s)}} dy$
3. B_t has stationary independent mean zero increments with $E[|B_t - B_s|^2] = d(t-s)$
4. $e^{\lambda \cdot B_t - \frac{1}{2}|\lambda|^2 t}$ is a martingale for any λ

Note that 1 does not depend on the basis: If $B_t^1, \ldots, B_t^2$ independent and $\mathcal{O}$ is orthogonal, then the coordinates of $\mathcal{O}B_t$ are independent Brownian motions   in fact

Theorem

Suppose X_1, X_2 independent and $\exists \theta \neq N\pi/2$ such that

$$X_1 \cos\theta + X_2 \sin\theta, \qquad -X_1 \sin\theta + X_2 \cos\theta \qquad \text{independent}$$

Then X_1, X_2 are Gaussians (Maxwell)

Dirichlet problem

Given a bounded open subset $G \subset \mathbf{R}^d$ and a continuous function $f : \partial G \to \mathbf{R}$ find a continuous function $u : \bar{G} \to R$ such that

$$\begin{cases} \Delta u = 0 & \text{in } G \\ u|_{\partial G} = f \end{cases}$$

$$\Delta u \stackrel{def}{=} \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = 2d \lim_{r \to 0} r^{-2} \left(\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} udS - u(x) \right)$$

Lemma

u harmonic in $G \Leftrightarrow u$ satisfies the mean value property: for all sufficiently small $r > 0$,

$$\frac{1}{|\partial S(r,x)|} \int_{\partial S(r,x)} udS = u(x)$$

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Proof.

Green's identity $\int_G v \Delta u dx = \int_G u \Delta v dx + \int_{\partial G} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \; dS$

$$G = \{\delta < x < r\}, \qquad v = \begin{cases} \frac{\log r - \log |x|}{\log r - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - r^{2-d}}{\delta^{2-d} - r^{2-d}} & d > 2 \end{cases}$$

let $\delta \downarrow 0$ □

B_t *d*-dimensional Brownian motion starting at $x \in G$

$$\tau_G = \inf\{t \geq 0 \ : \ B(t) \notin G\}$$

$$u(x) = E_x[f(B(\tau_G))]$$

"Theorem" If ∂G "nice" then u solves the Dirichlet problem

$$E_x[f(B(\tau_G))] = \int_{\partial G} f(y)\pi_G(x, dy), \quad \pi_G(x, \Gamma) = P_x(B(\tau_G) \in \Gamma), \quad \Gamma \subset \partial G$$

Example. $G = B(x, r), \quad \pi_G(x, \Gamma) = \frac{|\Gamma|}{|\partial S(x,r)|}, \qquad \Gamma \subset S(x, r)$

Brownian motion is invariant under rotations
$\therefore \pi_G(x, \cdot)$ is invariant under rotations

Proposition

G bounded open $\subset \mathbb{R}^d$, f bounded measurable on ∂G. Then $u(x) = E_x[f(B(\tau_G))]$ is harmonic in G.

Proof.

$B = B(x, r) \subset G \qquad \tau_B \leq \tau_G$

Strong Markov property: $u(B(\tau_S)) = E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]$

$$\begin{aligned} u(x) = E_x[f(B(\tau_G))] &= E_x[E_x[f(B(\tau_G)) \mid \mathcal{F}_{\tau_S}]] \\ &= E_x[u(B(\tau_S))] \\ &= \int_{\partial S} u(y)\pi_S(x, dy) \\ &= \frac{1}{|\partial S|}\int_{\partial S} u(y)dS \end{aligned}$$

So u satisfies the mean value property in G. □

$$a \in \partial G$$

To complete the proof that u solves the Dirichlet problem we need

$$\lim_{x \to a,\ x \in G} E_x[f(B(\tau_G))] = f(a)$$ It is not always true!

Proposition

If $\lim_{\substack{x \to a \\ x \in G}} P_x[\tau_G > \epsilon] = 0$, $\forall \epsilon > 0$ then for any bdd mble function $f : \partial G \to \mathbb{R}$ which is continuous at a, $\lim_{\substack{x \to a \\ x \in G}} E_x[f(B(\tau_G))] = f(a)$

Proof.

Need: $\lim_{x \to a,\ x \in G} P_x\left(|B(\tau_G) - x| < \delta\right) = 1$

$$\begin{aligned} P_x(|B(\tau_G) - x| < \delta) &\geq P_x\left(\sup_{0 \le t \le \epsilon} |B(t) - x| < \delta,\ \tau_G \le \epsilon\right) \\ &\geq P_x\left(\sup_{0 \le t \le \epsilon} |B(t) - x| < \delta\right) - P_x(\tau_G \le \epsilon) \\ &\to 1 \ \text{ as } \ x \to a, x \in G \ \text{ then } \ \epsilon \downarrow 0 \end{aligned}$$

Proposition

$a \in \partial G$ is *regular* if $P_a(\sigma_G = 0) = 1 \quad \sigma_G = \inf\{t > 0 \ : \ B(t) \notin G\}$
a regular $\Leftrightarrow \quad \lim_{x \to a,\ x \in G} E_x[f(B_{\tau_G})] = f(a)\ \forall f$ bdd mble, cont at a

Proof of $\Rightarrow$

Enough to prove $P_x(\sigma_G < \epsilon)$ lower semi-continuous in x
Then $\limsup_{\substack{x \to a \\ x \in G}} P_x(\sigma_G < \epsilon) \geq P_a(\sigma_G < \epsilon) = 1$ and $\sigma_G \geq \tau_G$
But $\int p(0, x, \delta, y) P_y(\exists s \in (0, \epsilon - \delta), B(s) \notin G)$ continuous
and $\uparrow P_x(\sigma_G < \epsilon)$ as $\delta \downarrow 0$

Examples

1. If ∂G is a smooth manifold near a then a is regular by LIL
2. If $\exists$ cone C of height $h > 0$ and vertex at a such that $C - \{a\} \subset \bar{G}^C$ then a is a regular (exterior cone condition)
3. $d \leq 2$ always, $d \geq 3$ $\exists$counterexamples

Application to recurrence/transience of Brownian motion

$$G = \{y \in \mathbf{R}^d \ : \ \delta < |y| < R\} \qquad f = \begin{cases} 0 & |y| = R \\ 1 & |y| = \delta \end{cases}$$

$$u(x) = E_x[f(B(\tau_G))] = P_x(\tau_\delta < \tau_R) = \begin{cases} \frac{\log R - \log |x|}{\log R - \log \delta} & d = 2 \\ \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} & d > 2 \end{cases}$$

Theorem In $d \geq 2$, Brownian motion does not visit a point

Proof.

$$P_x(\tau_0 < \tau_R) = \lim_{\delta \downarrow 0} P_x(\tau_\delta < \tau_R) = \lim_{\delta \downarrow 0} \frac{\log R - \log |x|}{\log R - \log \delta} = 0$$

□

Theorem In $d = 2$, Brownian motion is recurrent, ie. comes arbitrarily close to any point arbitrarily many times

Proof.

$$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} P_x(\tau_\delta < \tau_R) = \lim_{R\uparrow\infty} \frac{\log R - \log|x|}{\log R - \log\delta} = 1$$

□

Theorem In $d \geq 3$, Brownian motion wanders off to infinity

Proof.

$P_x(\tau_\delta < \infty) = \lim_{R\uparrow\infty} \frac{|x|^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}} = \left(\frac{|x|}{\delta}\right)^{2-d}$ if $|x| > \delta$

$P_x(\text{hit } |y| = \delta \text{ after time } t) = \int \frac{e^{-\frac{|x-y|^2}{2t}}}{(2\pi t)^{d/2}} P_y(\tau_\delta < \infty) dy \to 0$ as $t \to \infty$

$P_x(\liminf_{t\to\infty} |B(t)| > \delta) = 1$

$\delta \uparrow \infty \qquad \liminf_{t\to\infty} |B(t)| = \infty \qquad a.s.$

□

$\int_0^t B(s)dB(s)$

Based on experience with Riemann integrals

$$\int_0^t B(s)dB(s) = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(t_j^n)\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right)$$

for some choice of $t_j^n \in [\frac{j}{2^n}, \frac{j+1}{2^n}]$. Lets try two choices, the right and left endpoints.

$$L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j}{2^n})\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right)$$

$$R_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j+1}{2^n})\left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n})\right).$$

$$L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j}{2^n}) \left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n}) \right)$$

$$R_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} B(\frac{j+1}{2^n}) \left(B(\frac{j+1}{2^n}) - B(\frac{j}{2^n}) \right).$$

$$R_t - L_t = t$$

$$R_t + L_t = \lim_{n\to\infty} \sum_{j=0}^{\lfloor 2^n t\rfloor -1} (B^2(\frac{j+1}{2^n}) - B^2(\frac{j}{2^n})) = B^2(t).$$

$$L_t = \frac{1}{2}([L_t + R_t] - [R_t - L_t]) = \frac{1}{2}(B^2(t) - t) \qquad R_t = \frac{1}{2}(B^2(t) + t)$$

The choice of t_j^n matters! This is why Riemann told you to only integrate functions of bounded variation.

Which one is correct?

$\int_0^t BdB = \frac{1}{2}(B^2(t) - t)$ or $\frac{1}{2}(B^2(t) + t)$?

Or something else??

eg. midpoint rule gives $\int_0^t BdB = \frac{1}{2}B^2(t)$ which looks reasonable

Not really a mathematical question. A modeling question

Of all choices two have some special properties:

$L_t = \int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - t)$ is a martingale : Itô integral

Midpoint rule $\int_0^t B(s) \circ dB(s) = \frac{1}{2}B^2(t)$ looks like ordinary calculus : Stratonovich integral

We will *always* use the Itô integral and think of Stratonovich as a simple transformation of it which is sometimes useful in applications

(eg. Math finance: Itô, Math biology: Sometimes Stratonovich)

Definition: Progressively measurable

$\sigma(s,\omega)$ is called *progressively measurable* if

1. i. $\sigma(s,\omega)$ is $\mathcal{B}[0,\infty) \times \mathcal{F}$ measurable;
2. ii. For all $t \geq 0$, the map $[0,t] \times \Omega \to \mathbb{R}$ given by $\sigma(s,\omega)$ is $\mathcal{B}[0,t] \times \mathcal{F}_t$ measurable.

$\mathcal{B}[0,t]$ denotes the Borel σ-algebra on $[0,t]$.
Informally, $\sigma(s,\omega)$ is *nonanticipating*= uses information about ω contained in $\mathcal{F}_s$.

Definition: Simple Functions

$\sigma(s,\omega)$ is called *simple* if there exists a partition $0 \leq s_0 < s_1 < \cdots$ of $[0,\infty)$ and bounded random variables $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ such that $\sigma(s,\omega) = \sigma_j(\omega)$ for $s_j \leq s < s_{j+1}$.

Definition: Stochastic Integral for Simple Functions

Given such a $\sigma(s,\omega) = \sigma_j(\omega)$ for $s_j \leq s < s_{j+1}$, $\sigma_j(\omega) \in \mathcal{F}_{s_j}$ define

$$\int_0^t \sigma(s,\omega)dB(s) = \sum_{j=0}^{J(t)-1} \sigma_j(\omega)(B(s_{j+1})-B(s_j))+\sigma_{J(t)}(\omega)(B(t)-B(s_{J(t)}))$$

where $s_{J(t)} < t \leq s_{J(t)+1}$.

Basic properties

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale.
3. $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$.
4. $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale.

Lemma

Suppose that σ is progressively measurable and that for each $t \geq 0$,

$$E[\int_0^t \sigma^2(s,\omega)ds] < \infty.$$

Then there is a sequence σ_n of simple progressively measurable functions such that

$$E[\int_0^t (\sigma(s,\omega) - \sigma_n(s,\omega))^2 ds] \to 0.$$

Proof

We can assume that σ is bounded For if $\sigma_N = \sigma$ for $|\sigma| \leq N$ and 0 otherwise then $\sigma_N \to \sigma$ and $|\sigma_N - \sigma|^2 \leq 4|\sigma|^2$ so by the dominated convergence theorem $E[\int_0^t (\sigma - \sigma_N)^2 ds] \to 0$.

Proof.

Furthermore we can assume that σ is continuous in s

for if σ is bounded then $\sigma_h = h^{-1} \int_{t-h}^{t} \sigma ds$ are continuous progressively measurable and converge to σ as $h \to 0$. By the bounded convergence theorem

$$E[\int_0^t (\sigma - \sigma_h)^2 ds] \to 0$$

For σ continuous bounded and progressively measurable let

$$\sigma_n(s, \omega) = \sigma(\frac{\lfloor ns \rfloor}{n}, \omega)$$

These are progressively measurable, bounded and simple functions converging to σ and again by the bounded convergence theorem,

$$E[\int_0^t (\sigma - \sigma_n)^2 ds] \to 0$$

Theorem (Definition of the Itô Integral)

Let $\sigma(s,\omega)$ be progressively measurable and for each $t \geq 0$, $E[\int_0^t \sigma^2 ds] < \infty$. Let σ_n be simple functions with $E[\int_0^t (\sigma_n - \sigma)^2 ds] \to 0$ and set

$$X_n(t,\omega) = \int_0^t \sigma_n(s,\omega)dB(s).$$

Then

$$X(t,\omega) = \lim_{n\to\infty} X_n(t,\omega)$$

exists uniformly in probability, i.e. for each $T > 0$ and $\epsilon > 0$,

$$\lim_{n\to\infty} P(\sup_{0\leq t\leq T} |X_N(t,\omega) - X(t,\omega)| \geq \epsilon) = 0.$$

Furthermore the limit is independent of the choice of approximating sequence $\sigma_n \to \sigma$. The limit $X(t,\omega)$ is the Itô integral

$$X(t) = \int_0^t \sigma(s)dB(s)$$

Proof.

$X_n(t) - X_m(t) = \int_0^t (\sigma_n - \sigma_m) dB$ is a continuous martingale so by Doob's inequality

$$P(\sup_{0 \le t \le T} |X_n(t) - X_m(t)| \ge \epsilon) \le \epsilon^{-2} E[(X_n - X_m)^2(T)]$$
$$= \epsilon^{-2} E[\int_0^T (\sigma_n - \sigma_m)^2 ds]$$

So $X_n - X_m$ is uniformly Cauchy in probability and therefore there exists a progressively measurable X with

$$P(\sup_{0 \le t \le T} |X(t,\omega) - X_n(t,\omega)| \ge \epsilon) \stackrel{n\to\infty}{\to} 0 \qquad \epsilon > 0$$

If $\sigma'_n \stackrel{L^2}{\to} \sigma$ and $X'_n = \int_0^t \sigma'_n dB$, $P(\sup_{0 \le t \le T} |X_n - X'_n| \ge \epsilon) \to 0$ so that X_n and X'_n have the same limit. □

Basic properties of the Itô Integral

1. $\int_0^t (c_1\sigma_1 + c_2\sigma_2)dB = c_1 \int_0^t \sigma_1 dB + c_2 \int_0^t \sigma_2 dB$.
2. $\int_0^t \sigma dB$ is a continuous martingale.
3. $E[(\int_0^t \sigma(s,\omega)dB(s))^2] = E[\int_0^t \sigma^2(s,\omega)ds]$.
4. If $|\sigma| \leq C$ then $Z(t) = \exp\{\int_0^t \sigma dB - \frac{1}{2}\int_0^t \sigma^2 ds\}$ is a continuous martingale

proof

1. By construction
2. Continuity follows from the construction. To prove the limit is a martingale we have $E[X_n(t) \mid \mathcal{F}_s] = X_n(s)$ and $X_n \to X$ in L^2, therefore in L^1 as well. The L^1 limit of a martingale is a martingale.
3. $X_n^2(t) - \int_0^t \sigma_n^2(s)ds$ is a martingale $\xrightarrow{L^1} X^2(t) - \int_0^t \sigma^2(s)ds$
4. $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$ is a martingale so it suffices to show that $Z_n(t)$, $n = 1, 2, \ldots$ is a uniformly integrable family.

Proof.

to show that $Z_n(t) = \exp\{\int_0^t \sigma_n dB - \frac{1}{2}\int_0^t \sigma_n^2 ds\}$, $n = 1, 2, \ldots$ is a uniformly integrable family, it is enough to show that there is some fixed $C < \infty$ for which $E[(Z_N(t))^2] \leq C$.

$$\begin{aligned} E[(Z_N(t))^2] &= E[\exp\{2\int_0^t \sigma_n dB - \int_0^t \sigma_n^2 ds\}] \\ &\leq e^{Ct} E[\exp\{2\int_0^t \sigma_n dB - \frac{4}{2}\int_0^t \sigma_n^2 ds\}] \\ &= e^{Ct} \end{aligned}$$

□

A *stochastic integral* is an expression of the form

$$X(t,\omega) = \int_0^t \sigma(s,\omega)dB(s) + \int_0^t b(s,\omega)ds + X_0$$

where σ and b are progressively measurable with $E[\int_0^t \sigma^2(s,\omega)ds] < \infty$ and $\int_0^t |b(s,\omega)|ds < \infty$ for all $t \geq 0$, and $X_0 \in \mathcal{F}_0$ is the starting point

The stochastic differential

$$dX = \sigma dB + bdt$$

is shorthand for the same thing

For example the integral formula $\int_0^t B(s)dB(s) = \frac{1}{2}(B^2(t) - t)$ can be written in differential notation as

$$dB^2 = 2BdB + dt$$

What happens if $B^2(t)$ is replaced by a more general function $f(B(t))$?

Itô's Lemma

Let $f(x)$ be twice continuously differentiable. Then

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt$$

Proof

First of all we can assume without loss of generality that f, f' and f'' are all uniformly bounded, for if we can establish the lemma in the uniformly bounded case, we can approximate f by f_n so that all the corresponding derivatives are bounded and converge to those of f uniformly on compact sets.

Let $s = t_0 < t_1 < t_2 < \cdots < t_n = t$. We have

$$
\begin{aligned}
f(B(t)) - f(B(s)) &= \sum_{j=0}^{n-1} [f(B(t_{j+1})) - f(B(t_j))] \\
&= \sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \\
&\quad + \sum_{j=0}^{n-1} \frac{1}{2} f''(B(t_j))(B(t_{j+1}) - B(t_j))^2 \\
&\quad + \sum_{j=0}^{n-1} o\left((B(t_{j+1}) - B(t_j))^2\right).
\end{aligned}
$$

Let the width of the partition go to zero. By definition of the stochastic integral

$$
\sum_{j=0}^{n-1} f'(B(t_j))(B(t_{j+1}) - B(t_j)) \to \int_s^t f' dB.
$$

By the same argument as for the computation of the quadratic variation,

$$E\left[\left(\sum_{j=0}^{n-1}\frac{1}{2}f''(B(t_j))\left[(B(t_{j+1})-B(t_j))^2-(t_{j+1}-t_j)\right]\right)^2\right]\to 0.$$

Hence $\sum_{j=0}^{n-1}\frac{1}{2}f''(B(t_j))(B(t_{j+1})-B(t_j))^2\to\frac{1}{2}\int_s^t f''(B(u))du$ in L^2. The same argument shows that the last term goes to zero in L^2. So we have proved that

$$f(B(t)-f(B(s))=\int_s^t f'(B(u))dB(u)+\frac{1}{2}\int_s^t f''(B(u))du$$

which is Itô's formula.

1. In differential notation Itô's formula reads

$$df(B) = f'(B)dB + \frac{1}{2}f''(B)dt.$$

   The Taylor series is $df(B) = \sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(B)(dB)^n$. In normal calculus we would have $(dB)^n = 0$ if $n \geq 2$, but because of the finite quadratic variation of Brownian paths we have $(dB)^2 = dt$, while still $(dB)^n = 0$ if $n \geq 3$.

2. If the function f depends on t as well as $B(t)$, the formula is

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \frac{\partial f}{\partial x}(t, B(t))dB(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B(t))dt.$$

   The proof is about the same as the special case above.

3. If $B(t)$ is a d-dimensional Brownian motion and $f(t, x)$ is a function on $[0, \infty) \times \mathbf{R}^d$ which has one continuous derivative in t and two continuous derivatives in x, then the formula reads

$$df(t, B(t)) = \frac{\partial f}{\partial t}(t, B(t))dt + \nabla f((t, B(t)) \cdot dB(t) + \frac{1}{2}\Delta f(t, B(t))dt.$$