Einstein's derivation of Brownian transition density

Particle starts at $0 \in \mathbf{R}^3$ and is pushed around by tiny molecular bombardments.

$f(t, x)dx = P(X_t \in dx) = \lim_{h \to 0} h^{-3} P(X_t \in \text{a box of side length h around x})$.

$$p(s, x, t, y)dy = P(X_t \in dy \mid X_s = x).$$

$$f(t + \tau, x) = \int f(x - y, t)p(x - y, t, x, t + \tau)dy.$$

homogeneity in space and time $p(s, x, t, y) = p(t - s, y - x)$.

$$
\begin{aligned}
f(t+\tau,x) &= \int (f(t,x) - y\cdot\nabla f(t,x) + \frac{1}{2} y\cdot D^2 f(t,x) y + \cdots) p(\tau,y)dy \\
&= f(t,x)\int p(\tau,y)dy - \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(t,x)\int y_i p(\tau,y)dy \\
&+ \frac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2 f}{\partial x_i \partial x_j}(t,x)\int y_i y_j p(\tau,y)dy + \cdots .
\end{aligned}
$$

$$\int p(\tau,y)dy = 1.$$

symmetry $\int y_i p(\tau,y)dy = 0$ and $\int y_i y_j p(\tau,y)dy = 0 \qquad i \neq j.$

Influence of molecular bombardment in any two nonoverlapping intervals of time is independent

Variance should grow linearly, like the sum of independent random variables

$$Var(X_1 + \cdots + X_N) \simeq CN$$

$$\int y_i^2 p(\tau, y) dy = D\tau.$$

Letting $\tau \to 0$ get heat equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} D \Delta f.$$

With the obvious initial condition $f(x, 0) = \delta_0$ this has the well known solution

$$f(t, x) = \frac{e^{-\frac{|x|^2}{2Dt}}}{(2\pi Dt)^{3/2}}.$$

Markov processes

A process $X_t,\ t \geq 0$ is called a Markov process if for any function g and any $t \geq s$,

$$E[g(X_t) \mid X_u,\ 0 \leq u \leq s] = E[g(X_t) \mid X_s].$$

Process determined by initial distr $P(X_0 \in A)$ and the transition probs

$$p(s, x, t, A) = P(X_t \in A \mid X_s = x) \qquad s < t$$

$$P(X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n)$$
$$= \int_{A_{n-1}} \cdots \int_{A_1} \int P(X_0 \in dx_0) p(0, x_0, t_1, dx_1) \cdots p(t_{n-1}, x_{n-1}, t_n, A_n)$$

Chapman-Kolmogorov equations

$$p(s, x, t, A) = \int p(s, x, u, dy) p(u, y, t, A) \qquad \text{for } s \leq u \leq t.$$

Example. Brownian motion $p(s, x, t, dy) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} dy$

Gaussian measures

Definition

Let $(E, \mathcal{E}, \mu)$ be σ-finite, separable.
There exists a Gaussian family $\{X(f)\}_{f \in L^2(E,\mathcal{E},\mu)}$ satisfying

1. $f \mapsto X(f)$ is linear
2. $E[X(f)] = 0$, $E[|X(f)|^2] = \|f\|^2_{L^2(E,\mathcal{E},\mu)}$

We can write $X(A) = X(1_A)$, $A \in \mathcal{E}$ to get a random measure

Example: Brownian motion

$X_t - X_s = X([s,t))$ is a Gaussian measure
intensity $\mu =$ Lebesgue measure

$$X(f) = \int f dB \quad (1)$$

Gaussian measures

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Proof.

Let $\{e_n\}_{n=1,2,3\ldots}$ be an orthonormal basis of $L^2(E, \mathcal{E}, \mu)$ and X_n, $n = 1, 2, \ldots$ be iid $\mathcal{N}(0, 1)$.

$$X(f) = \sum_{n=1}^{\infty} \langle f, e_n \rangle X_n$$

□

Functions of finite variation

$f : \mathbb{R}_+ \to \mathbb{R}$ right continuous

$\Delta = 0 = t_0 < t_1 < \cdots < t_n = t$ subdivision of $[0, t]$

$|\Delta| = \sup_i |t_{i+1} - t_i| =$ mesh size

f is of finite variation if for each $t < \infty$,
$\|f\|_{TV,[0,t]} = \sup_\Delta \sum_i |f(t_{i+1}) - f(t_i)| < \infty$

Proposition

1. *A function of finite variation is the difference of two monotone increasing functions*
2. $\mu([0, t]) = f(t)$ *provides a 1-1 correspondence between measures on* $\mathbb{R}_+$ *and functions of finite variation*
3. $\int_0^\infty g(t)df(t) = \int g d\mu$ *is the Riemann-Stieltjes integral*
4. *A function of finite variation is differentiable almost everywhere*

Quadratic variation

Definition

A stochastic process X_t, $t \geq 0$ has finite quadratic variation if there exists a finite process $\langle X, X \rangle_t$, $t \geq 0$s.t. for each $t < \infty$ and each sequence $\{\Delta_n\}_{n=1,2,\ldots}$ of subdivisions of $[0, t]$ with $|\Delta_n| \to 0$,

$$\lim_{n\to\infty} \sum_i |X_{t_{i+1}} - X_{t_i}|^2 \stackrel{prob}{=} \langle X, X \rangle_t$$

The process $\langle X, X \rangle_t$, $t \geq 0$ is non-decreasing.
It is called the *quadratic variation* of X.

Recall $\lim_{n\to\infty} X_n \stackrel{prob}{=} X$ if $P(|X_n - X| \geq \epsilon) \to 0$ for each $\epsilon > 0$

Quadratic variation

Theorem

Let X be a Gaussian measure with intensity μ on $(E, \mathcal{E})$ Let $A \in \mathcal{E}$ with $\mu(A) < \infty$ Let $\{A_k^n\}_{n=1,2,\ldots}$ be finite partitions of A such that $\sup_k \mu(A_k^n) \to 0$ as $n \to \infty$ Then

$$\lim_{n\to\infty} \sum_k |X(A_k^n)|^2 = \mu(A)$$

in $L^2(\Omega, \mathcal{F}, P)$

Proof.

$\{X(A_K^n)\}_k$ independent $\mathcal{N}(0, |\mu(A_k^n)|)$ so

$$E[|\textstyle\sum_k X^2(A_K^n) - \mu(A_k^n)|^2] = \sum_k E[|X^2(A_K^n) - \mu(A_k^n)|^2]$$
$$= 2\textstyle\sum_k |\mu(A_k^n)|^2 \leq 2\mu(A) \sup_k \mu(A_k^n)$$

□

Quadratic variation

Theorem

Let X be a Gaussian measure with intensity μ on $(E, \mathcal{E})$ Let $A \in \mathcal{E}$ with $\mu(A) < \infty$
Let $\{A_k^n\}_{n=1,2,\ldots}$ be finite partitions of A such that $\sup_k \mu(A_k^n) \to 0$ as $n \to \infty$
Then

$$\lim_{n\to\infty} \sum_k |X(A_k^n)|^2 = \mu(A)$$

in $L^2(\Omega, \mathcal{F}, P)$

Brownian motion

Brownian motion is Gaussian measure with intensity dt

$$\langle B, B\rangle_t = t \tag{2}$$

Quadratic variation

Note that if $X_n \stackrel{prob}{\to} X$ one can always choose a (non-random) subsequence such that $X_n \stackrel{prob}{\to} X$
So one can choose partitions so that

$$\lim_{n\to\infty} \sum_i |X_{t_{i+1}} - X_{t_i}|^2 \stackrel{a.s.}{=} \langle X, X \rangle_t$$

For Brownian motion it turns out that any sequence $\Delta_n \subset \Delta_{n+1}$ gives a.s. convergence

Proposition

With probability one Brownian motion B_t, $t \geq 0$ is not of finite variation in any interval

Proof.

Let f be any continuous function on $[0, t]$

$$\sum_i |f_{t_{i+1}} - f_{t_i}|^2 \leq \max_i |f_{t_{i+1}} - f_{t_i}| \sum_i |f_{t_{i+1}} - f_{t_i}|$$

Since $\max_i |f_{t_{i+1}} - f_{t_i}| \to 0$, if

$$\langle f, f \rangle_t = \lim \sum_i |f_{t_{i+1}} - f_{t_i}|^2 > 0$$

then

$$\lim \sum_i |f_{t_{i+1}} - f_{t_i}| = \|f\|_{TV,[0,t]} = \infty$$

and if $\|f\|_{TV,[0,t]} < \infty$ then $\langle f, f \rangle_t = 0$ □

Proposition

With probability one Brownian motion B_t, $t \geq 0$ is not locally Hölder of order α for any $\alpha > 1/2$

Proof.

Let f be any continuous function on $[0, t]$ s.t. for some $0 \leq a < b \leq t$ and some $\alpha > 1/2$, for all $a \leq s, t \leq b$,

$$|f_t - f_s| \leq k|t - s|^\alpha$$

Then

$$\sum_i |f_{t_{i+1}} - f_{t_i}|^2 \leq k^2(b - a) \max_i |t_{i+1} - t_i|^{2\alpha - 1}$$

□

Theorem. (Paley, Wiener, Zygmund 33)

Brownian motion is nowhere differentiable with probability one

Proof. (Dvoretsky, Erdös, Kakutani 61)

Suppose that $B(t)$ was differentiable at a point $s \in [0,1]$.

Then $\exists \epsilon > 0$ and an integer $\ell \geq 1$ such that

$$|B(t) - B(s)| \leq \ell(t-s) \qquad \text{for} \quad 0 < t - s < \epsilon.$$

Choose an integer $n > \ell$ large enough so that

$$s \leq \frac{i}{n} < \frac{i+1}{n} < \frac{i+2}{n} < \frac{i+3}{n} < s + \epsilon \qquad \text{where} \quad i = \lfloor ns \rfloor + 1.$$

Then

$$|B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \qquad \text{for} \quad j = i+1, i+2, i+3.$$

Proof.

Therefore the event that $B(t)$ is differentiable at some point is contained in the set

$$B = \bigcup_{\ell \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} \bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\}.$$

We show $P(B) = 0$ as follows.

$$P\left(\bigcap_{n \geq m} \bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\} \right)$$
$$\leq \liminf_{n \to \infty} P\left(\bigcup_{0 \leq i \leq n+1} \bigcap_{i \leq j \leq i+3} \left\{ |B(\frac{j}{n}) - B(\frac{j-1}{n})| < \frac{7\ell}{n} \right\} \right)$$

Proof.

$$
\begin{aligned}
&P\left(\bigcap_{n\geq m}\bigcup_{0\leq i\leq n+1}\bigcap_{i\leq j\leq i+3}\left\{|B(\tfrac{j}{n})-B(\tfrac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty} P\left(\bigcup_{0\leq i\leq n+1}\bigcap_{i\leq j\leq i+3}\left\{|B(\tfrac{j}{n})-B(\tfrac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty}\sum_{i=1}^{n+1} P\left(\bigcap_{i\leq j\leq i+3}\left\{|B(\tfrac{j}{n})-B(\tfrac{j-1}{n})|<\frac{7\ell}{n}\right\}\right)\\
&\leq \liminf_{n\to\infty} n\left[P\left(|B(\tfrac{1}{n})|<\frac{7\ell}{n}\right)\right]^3\\
&= \liminf_{n\to\infty} n\left[P\left(|B(1)|<\frac{7\ell}{\sqrt{n}}\right)\right]^3 = \liminf_{n\to\infty} n\left[\frac{7\ell}{\sqrt{n}}\right]^3 = 0
\end{aligned}
$$

□

Theorem Brownian motion is not Hölder of order 1/2

this follows from

Modulus of continuity (P.Levy)

With probability one,

$$\limsup_{\epsilon \longrightarrow 0} \sup_{\substack{0 \leq s < t \leq 1 \\ t-s<\epsilon}} \frac{|B_t - B_s|}{\sqrt{2\epsilon \log \epsilon^{-1}}} = 1$$

Lemma

$$\frac{x}{x^2+1} e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{y^2}{2}} dy \leq x^{-1} e^{-\frac{x^2}{2}} \qquad x > 0$$

$$\int_x^\infty e^{-\frac{y^2}{2}} dy \leq x^{-1} \int_x^\infty y e^{-\frac{y^2}{2}} dy = x^{-1} e^{-\frac{x^2}{2}}$$

Proof.

$$x^{-2} \int_x^\infty e^{-\frac{y^2}{2}} dy \geq \int_x^\infty y^{-2} e^{-\frac{y^2}{2}} dy = x^{-1} e^{-\frac{x^2}{2}} - \int_x^\infty e^{-\frac{y^2}{2}} dy$$

Proof of $\limsup_{\epsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s<\epsilon}} \frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}} \ge 1$

let $\delta > 0$, $A_n = \{\max_{1\le k\le 2^n} |B_{\frac{k}{2^n}} - B_{\frac{k-1}{2^n}}| \le (1-\delta)h(2^{-n})\}$,
$h(t) = \sqrt{2t\log t^{-1}}$

$$P(A_n) \le (1 - 2\int_{(1-\delta)\sqrt{2\log 2^n}} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy)^{2^n} \quad \textit{independent increments}$$

$$\le e^{-Cn^{-1/2}2^{n(1-(1-\delta)^2)}} \qquad \textit{by lemma}$$

so $\sum_{n=1}^{\infty} P(A_n) < \infty$. By the Borel-Cantelli lemma, almost every ω is in at most finitely many A_n.i.e.

$$\limsup_{\epsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-t-s<\epsilon}} \frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}} \ge 1-\delta$$

now let $\delta \downarrow 0$

Note: This proves that Brownian motion is not Hölder of order $\alpha \ge 1/2$

Proof of $\limsup_{\epsilon\to 0} \sup_{\substack{0\le s<t\le 1\\ t-s<\epsilon}} \frac{|B_t-B_s|}{\sqrt{2\epsilon\log\epsilon^{-1}}} \le 1$

let $\delta > 0$ and choose $\epsilon > 0$ so that $(1+\epsilon)^2(1-\delta) > 1+\delta$
let

$$B_n = \{\max_{i,j\in K} \frac{|B_{j/2^n} - B_{i/2^n}|}{h(k/2^n)} \ge 1+\epsilon\}$$

$$K = \{0 \le i < j < 2^n, 0 < k = j - i \le 2^{n\delta}\}$$

$$\begin{aligned} P(B_n) &\le \sum_K \frac{2}{\sqrt{2\pi}} \int_{(1+\epsilon)\sqrt{\log(k^{-1}2^n)}}^{\infty} e^{-\frac{y^2}{2}} dy \\ &\le C \sum_K [\log(k^{-1}2^n)]^{-1/2} e^{-(1+\epsilon)^2 \log(k^{-1}2^n)} \quad \text{lemma} \\ &\le C 2^{-n(1-\delta)(1-\epsilon)^2} \sum_K [\log(k^{-1}2^n)]^{-1/2} \quad k^{-1} \ge 2^{-n\delta} \end{aligned}$$

$$B_n = \{\max_{i,j \in K} \frac{|B_{j/2^n} - B_{i/2^n}|}{h(k/2^n)} \geq 1 + \epsilon\}$$

$$K = \{0 \leq i < j < 2^n, 0 < k = j - i \leq 2^{n\delta}\}$$

$$P(B_n) \leq C2^{-n(1-\delta)(1-\epsilon)^2} \sum_K [\log(k^{-1}2^n)]^{-1/2}$$

$$\leq Cn^{-1/2}2^{n((1+\delta)-(1-\delta)(1+\epsilon)^2)} \quad |K| \leq 2^{n(1+\delta)}, \log(k^{-1}2^n) \geq \log 2^{n(1-\delta)}$$

summable so with probability one there is an N s.t. for $n > N$, for $i, j \in K$,

$$|B_{j/2^n} - B_{i/2^n}| < (1 + \epsilon)h(k/2^n)$$

Let $\gamma > 0$. Pick N large so that $\sum_{m=n+1}^{\infty} h(2^{-m}) \le \gamma h(2^{-(n+1)(1-\delta)})$, $n \ge N$

Suppose that $t = i2^{-n} + 2^{-n_1} + 2^{-n_2} + \cdots$ with $N \le n < n_1 < n_2 < \cdots$ and $i \in K$,

$$|B_t - B_{i/2^n}| \le (1+\epsilon) \sum_{m=n+1}^{\infty} h(2^{-m}) \le (1+\epsilon)\gamma h(2^{-(n+1)(1-\delta)})$$

now suppose we have $0 \le s < t \le 1$ and the special n so that

$$2^{-(n+1)(1-\delta)} \le t - s < 2^{-n(1-\delta)}$$

has $n \ge N$ then we can write

$$\begin{aligned} |B_t - B_s| &\le |B_{i/2^n} - B_s| + |B_{j/2^n} - B_{i/2^n}| + |B_t - B_{j/2^n}| \\ &\le 2(1+\epsilon)\gamma h(2^{-(n+1)(1-\delta)}) + (1+\epsilon)h((j-i)2^{-n}) \\ &\le (2(1+\epsilon)\gamma + 1 + \epsilon)h(t-s) \qquad \textit{if } t-s \textit{ small enough} \end{aligned}$$

let $\delta \downarrow 0$ and then $\gamma \downarrow 0$