Brownian motion as the limit of random walks

$X_1, X_2, \ldots$ iid Bernoulli $P(X_i = 1) = P(X_i = -1) = 1/2$

$$S_n = X_1 + \cdots + X_n$$

$B_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor tn \rfloor}$ Takes steps $\pm \frac{1}{\sqrt{n}}$ at times $\frac{1}{n}, \frac{2}{n}, \ldots$

Or $\bar{B}_n(t) =$ polygonalized version. Almost the same but continuous

$$B_n(t) \xrightarrow{n \to \infty} \text{Brownian motion } B(t)$$

What does it mean for stochastic processes to converge?

$$\text{dist}(B_n(t_1), \ldots, B_n(t_k)) \to \text{dist}(B(t_1), \ldots, B(t_k)) \qquad k = 1, 2, 3, \ldots$$

Convergenence of finite dimesional distributions
Immediate from (multidimensional) central limit theorem
Same for $\bar{B}_n(t)$

P_n = measure on $C[0, T]$ corresponding to $\bar{B}_n(t)$, $0 \leq t \leq T$

Invariance principle (Donsker's Theorem)

$$P_n \Rightarrow P$$

Much stronger than convergence of finite dimensional distributions

Examples

1. $$dist(\max_{0\leq m\leq n} \frac{1}{\sqrt{n}} S_m) \rightarrow dist(\sup_{0\leq t\leq 1} B(t))$$

2. $$dist(n^{-1-\frac{k}{2}} \sum_{m=1}^{n} S_m^k) \rightarrow dist(\int_0^1 B^k(t)dt)$$

For a proof see Billingsley "Convergence of Probability Measures"

Brownian motion with variance σ^2 and drift b as the limit of random walks

$X_n(t)$ jumps $\frac{1}{\sqrt{n}}\sigma + \frac{1}{n}b$ or $-\frac{1}{\sqrt{n}}\sigma + \frac{1}{n}b$ with probabilities $1/2$ at times $\frac{1}{n}, \frac{2}{n}, \ldots$

$$X_n(t) - b\frac{\lfloor nt \rfloor}{n} = \sigma B_n(t)$$

$$X_n(t) \to \sigma B(t) + bt$$

General local diffusivity $\sigma^2(t, x)$ and drift $b(t, x)$

$X_n(t)$ jumps

$$\frac{1}{\sqrt{n}}\sigma(\tfrac{i}{n}, X_n(\tfrac{i}{n})) + \frac{1}{n}b(\tfrac{i}{n}, X_n(\tfrac{i}{n})) \quad \text{or} \quad -\frac{1}{\sqrt{n}}\sigma(\tfrac{i}{n}, X_n(\tfrac{i}{n})) + \frac{1}{n}b(\tfrac{i}{n}, X_n(\tfrac{i}{n}))$$

with probabilities $1/2$ at times $\frac{i}{n}$, $i = 1, 2, \ldots$ $\quad X_n(t) \to X(t)$

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$$

But how to prove it?

Here's another proof that random walks converge to Brownian motions, which does generalize

Recall $B_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor tn \rfloor}$ where $S_n = X_1 + \cdots + X_n$

Let $f \in C^2$

$$f(B_n(t)) = \frac{1}{n} \sum_{i=0}^{\lfloor tn \rfloor - 1} L_n f(B_n(\frac{i}{n})) = \text{ Martingale}$$

$$L_n f(x) = \frac{1}{2} n(f(x + n^{-1/2}) - 2f(x) + f(x - n^{-1/2}))$$

$$L_n f(x) \to \frac{1}{2} f''(x)$$

$$\frac{1}{n} \sum_{i=0}^{\lfloor tn \rfloor - 1} L_n f(B_n(\frac{i}{n})) \to \frac{1}{2} \int_0^t f''(B(s)) ds$$

$$f(B(t)) - \frac{1}{2} \int_0^t f''(B(s)) ds = \text{ martingale } \quad \Rightarrow \quad B(t) \text{ Brownian motion}$$

Really one needs to show that P_n are precompact as a set of probability measures. It is similar to the proof that Brownian motion is continuous, but you just use the martingale formulation directly. The details are long, but the final result is

Theorem

Suppose that

1. $n \int_{|y-x| \leq 1} (y_i - x_i)(y_j - x_j) p_{1/n}(x, dy) \to a_{ij}(x)$ uniformly on compact sets
2. $n \int_{|y-x| \leq 1} (y_i - x_i) p_{1/n}(x, dy) \to b_i(x)$ uniformly on compact sets
3. $n p_{1/n}(x, B(x, \epsilon)^C) \to 0$ uniformly on compact sets, for each $\epsilon > 0$

where $a(x)$ and $b(x)$ are continuous. Suppose that we have weak uniqueness for the stochastic differential equation

$$dX = \sigma(X) dB + b(X) dt$$

and let P denote the measure on $C[0, T]$ corresponding to $X(t)$, $0 \leq t \leq T$. Then $P_n \Rightarrow P$

Similarly

Theorem

Suppose that $\sigma(t,x)$ and $b(t,x)$ are locally bounded measurable functions, continuous in x for each $t \geq 0$ and one has weak uniqueness for the stochastic differential equation

$$dX = \sigma(X)dB + b(X)dt, \qquad X_0 = x$$

Let $P_x^{a,b}$ denote the measure on $C[0,T]$ corresponding to $X(t)$, $0 \leq t \leq T$, where $a = \sigma\sigma^T$. Suppose that $\sigma_n(t,x)$ and $b_n(t,x)$ are measurable and locally bounded uniformly in n, and that

$$\lim_{n\to\infty} \int_0^T \sup_{|x|\leq R} \{\|a_n(s,x) - a(s,x)\| + |b_n(s,x) - b(s,x)|\}\, ds = 0.$$

Then

$$P_x^{a_n,b_n} \Rightarrow P_x^{a,b}.$$

Binomial model

$S_n =$ price of asset at time n

$P(S_{n+1} = uS_n) = p$ $P(S_{n+1} = dS_n) = 1 - p = q$ Often $d = 1/u$

$r =$ interest rate $0 < d < 1 + r < u$

European call option has strike price K at time 1, $dS_0 < K < uS_0$

It means it is worth $uS_0 - K$ if the stock goes up, and 0 if the stock goes down

Question: How much is it worth today?

Let's see what you could do with the stock and money market

$X_n =$ wealth at time n

$\Delta_n =$ stock held at time n

Time 0: $\Delta_0 S_0$ stock, $X_0 - \Delta_0 S_0$ cash

At time 1 it is worth: $\Delta_0 S_1 + (1+r)X_0 - \Delta_0 S_0$

ie. it is worth $\Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0)$ if the stock goes up and $\Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0)$ if the stock goes down

Suppose we take
$\Delta_0 u S_0 + (1+r)(X_0 - \Delta_0 S_0) = uS_0 - K$,
$\Delta_0 d S_0 + (1+r)(X_0 - \Delta_0 S_0) = 0$

ie. $\Delta_0 = \frac{uS_0 - K}{(u-d)S_0}$, $X_0 = \frac{1+r-d}{u-d}(1+r)^{-1}(uS_0 - K)$

Suppose you have X_0 .
If the option costs more than X_0, you won't buy it, because you can do better with the stock and money market strategy just described

If the option costs less than X_0, it is a better deal than just described, so everyone would short the stock to buy it, and the stock would go down.

So as long as the market is efficient = no arbitrage opportunities, the option is worth

$$V_0 = (1+r)^{-1}\tilde{p}(uS_0 - K) \qquad \tilde{p} = \frac{1+r-d}{u-d}$$

If the option was worth V_1 at time 1 then you can do the computation again to get

$$V_0 = (1+r)^{-1}\tilde{E}[V_1]$$

where $\tilde{p}$ is the probability of the stock going up and $1-\tilde{p}$ is the probability of the stock going down

IT DOESN'T DEPEND ON p !!!!!

$(1+r)^{-n}V_n$ is a martingale wrt the "risk neutral" measure $\tilde{P}$

$$V_0 = (1+r)^{-n}\tilde{E}[V_n]$$

Limit of small transaction periods

For each n, consider a binomial model $S_n(t)$ with

$$\begin{aligned} u_n &= 1 + \frac{\sigma}{\sqrt{n}} \\ d_n &= 1 - \frac{\sigma}{\sqrt{n}} \\ r_n &= 0 \\ \text{period} &= \frac{1}{n} \end{aligned}$$

$$\tilde{p} = \frac{1 + r_n - d_n}{u_n - d_n} = \frac{1}{2} = 1 - \tilde{p}$$

At each time $1/n$, $X_n = \log S_n$ jump up $\log u_n$ or down $\log d_n$,each with probability $1/2$

$$\begin{aligned} nE[X_{n+1} - X_n \mid X_n = x] &= \frac{1}{2} n(\log u_n - \log d_n) \\ nE[(X_{n+1} - X_n)^2 \mid X_n = x] &= \frac{1}{2} n((\log u_n)^2 + (\log d_n)^2) \end{aligned}$$

$$\log u_n = \log(1 + \frac{\sigma}{\sqrt{n}}) = \frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + o(n^{-1})$$

$$\log d_n = \log(1 - \frac{\sigma}{\sqrt{n}}) = -\frac{\sigma}{\sqrt{n}} - \frac{\sigma^2}{2n} + o(n^{-1})$$

$$\begin{aligned} nE[X_{n+1} - X_n \mid X_n = x] &= \frac{1}{2} n(\log u_n + \log d_n) = -\frac{\sigma^2}{2} + o(1) \\ nE[(X_{n+1} - X_n)^2 \mid X_n = x] &= \frac{1}{2} n((\log u_n)^2 + (\log d_n)^2) = \sigma^2 \end{aligned}$$

Hence

$$X_n(t) \to X(t) \qquad dX = \sigma dB - \frac{1}{2}\sigma^2 dt$$

$$S_n(t) \to S(t) = e^{\sigma B_t - \frac{1}{2}\sigma^2 t} \qquad \text{Geometric Brownian motion}$$

European option pays $V(S(T))$ at time T

Value today $= V(t, S(t)) = e^{-r(T-t)} E[V(S(T)) \mid \mathcal{F}_t]$

Plug into normal density $\Rightarrow$ Black-Scholes formula

$$\begin{aligned} V(t, S_t) &= S_t \Phi\left(\frac{\log\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &\quad - e^{-r(T-t)} K \Phi\left(\frac{\log\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \end{aligned}$$

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$$

Brownian local time

Definition

$$L_t(x) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1(|B_s - x| \leq \epsilon) ds$$

To prove that it exists, and is continuous

$$f_\epsilon(y) = \begin{cases} |y|, & |y| \geq \epsilon \\ \frac{1}{2}\epsilon^{-1}(y^2 + \epsilon^2), & |x| \leq \epsilon \end{cases}$$

Then $\frac{1}{2} f_\epsilon''(y) = \frac{1}{2\epsilon} 1(|y| \leq \epsilon)$

$$f_\epsilon(B_t) - \int_0^t f_\epsilon'(B_s) dB_s = \frac{1}{2\epsilon} \int_0^t 1(|B_s - x| \leq \epsilon) ds$$

$$f_\epsilon(B_t) \to |B_t|$$

$$
\begin{aligned}
&E[(\int_0^t (f_\epsilon'(B_s) - \operatorname{sgn}(B_s))dB_s)^2] \\
&= E[\int_0^t (f_\epsilon'(B_s) - \operatorname{sgn}(B_s)))^2 ds] \\
&= E[\int_0^t (\epsilon^{-1}(B_s) - \operatorname{sgn}(B_s))^2 1(|B_s| \le \epsilon) ds] \\
&\le E[\int_0^t 1(|B_s| \le \epsilon) ds] \\
&\to 0 \qquad \text{as} \quad \epsilon \to 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{2\epsilon}\int_0^t 1(|B_s - x| \le \epsilon) ds &= f_\epsilon(B_t) - \int_0^t f_\epsilon'(B_s) dB_s \\
&\to |B_t| - \int_0^t \operatorname{sgn}(B_s) dB_s
\end{aligned}
$$

Same proof gives

Tanaka's formula's

$$|B_t - x| - \int_0^t \operatorname{sgn}(B_s - x)dB_s = L_t(x)$$

$$(B_t - x)_+ - \int_0^t 1(B_s \geq x)dB_s = \frac{1}{2}L_t(x)$$

$L_t(x)$ is continuous in t and x, nondecreasing in t

$\int_0^t \operatorname{sgn}(B_s - x)dB_s = \tilde{B}_t$

Lemma

Let $x(t)$ be a continuous function on $[0, \infty)$ with $x(0) = 0$.There exists a unique continuous function $a(t)$ on $[0, \infty)$ such that $a(0) = 0$, $a(t)$ is nondecreasing,$a(t)$ is increasing only on $\{t : x(t) + a(t) = 0\}$ in the sense that

$$\int_0^\infty 1(x(s) + a(s) > 0)da = 0$$

and such that

$$x(t) + a(t) \geq 0, \qquad t \geq 0$$

Proof

Let

$$a(t) = 0 \vee \sup_{0 \leq s \leq t} \{-x(s)\}$$

Then $x(t) \geq 0 \wedge \inf_{0 \leq s \leq t} x(s) = -a(t)$, $a(0) = 0$, $a(t)$ nondecreasing, continuous.

Proof

$$a(t) = 0 \vee \sup_{0 \le s \le t} \{-x(t)\}$$

Suppose $(t_1, t_2) \subset \{t \ge 0 : x(t) + a(t) > \epsilon\}$

$$-x(s) = a(s) - (x(s) + a(s)) \le a(t_2) - \epsilon, \qquad t_1 \le s \le t_2$$

so

$$a(t_2) = a(t_1)$$

Proof.

Uniqueness: Suppose $a_1(t)$ and $a_2(t)$ both do the job

Suppose $\exists\tau$,

$$x(\tau) + a_1(\tau) > x(\tau) + a_2(\tau)$$

$$\sigma = \sup\{0 \leq s < \tau : x(s) + a_1(s) = x(s) + a_2(s)\}$$

$$x(t) + a_1(t) > x(t) + a_2(t), \qquad \sigma < t \leq \tau$$

$$\begin{aligned} x(t) + a_2(t) \geq 0 \quad &\Rightarrow \quad x(t) + a_1(t) > 0 \qquad \sigma < t \leq \tau \\ &\Rightarrow \quad a_1(\tau) = a_1(\sigma) \\ &\Rightarrow \quad [x(\tau) + a_1(\tau)] - [x(\tau) - a_2(\tau)] \\ &\qquad = [x(\sigma) + a_1(\sigma)] - [x(\sigma) - a_2(\sigma)] = 0 \end{aligned}$$

□

$$\sup_{0\le s\le t} B_s = 0 \vee \sup_{0\le s\le t} \{-(-B_s\}$$

$$\text{Lemma} \;\Rightarrow\; \sup_{0\le s\le t} B_s - B_t \ge 0$$

$$\text{Tanaka} \;\Rightarrow\; -\tilde{B}_t + L_t(x) = |B_t| \ge 0$$

By uniqueness

$$L_t(x) \stackrel{dist}{=} \sup_{0\le s\le t} B_s \qquad \sup_{0\le s\le t} B_s - B_t \stackrel{dist}{=} |B_t|$$