Martingale representation theorem

$\Omega = C[0, T]$, $\mathcal{F}_T$ = smallest σ-field with respect to which B_s are all measurable, $s \leq T$, P the Wiener measure , B_t = Brownian motion

M_t square integrable martingale with respect to $\mathcal{F}_t$

Then there exists $\sigma(t, \omega)$ which is

1. progressively measurable
2. square integrable
3. $\mathcal{B}([0, \infty)) \times \mathcal{F}$ mble

such that

$$M_t = M_0 + \int_0^t \sigma(s) dB_s$$

Lemma

$\mathcal{A} =$ set of all linear combinations of random variables of the form

$$e^{\int_0^T h dB - \frac{1}{2}\int_0^T h^2 dt}, \qquad h \in L^2([0, T])$$

$\mathcal{A}$ is dense in $L^2(\Omega, \mathcal{F}_T, P)$

Proof

Suppose $g \in L^2(\Omega, \mathcal{F}_T, P)$ is orthogonal to all such functions

We want to show that $g = 0$

By an easy choice of simple functions h we find that for any $\lambda_1, \ldots, \lambda_n \in R$ and $t_1, \ldots, t_n \in [0, T]$,

$$E^P[g e^{\lambda_1 B_{t_1} + \cdots + \lambda_n B_{t_n}}] = 0$$

lhs real analytic in λ and hence has an analytic extension to $\lambda \in \mathbb{C}^n$

Since $E^P[ge^{\lambda_1 B_{t_1}+\cdots+\lambda_n B_{t_n}}]$ is analytic and vanishes on the real axis, it is zero everywhere. In particular

$$E^P[ge^{i(y_1 B_{t_1}+\cdots+y_n B_{t_n})}] = 0$$

Suppose $\phi \in C_0^\infty(\mathbb{R}^n)$

$$\hat{\phi}(y) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \phi(x) e^{-ix\cdot y} dx$$

Fourier inversion:

$$\phi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\phi}(y) e^{ix\cdot y} dy$$

$$E^P[g\phi(B_{t_1},\ldots,B_{t_n})] = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{\phi}(y) E^P[e^{iy_1 B_{t_1}+\cdots+y_n B_{t_n})}] dy = 0$$

Hence g is orthogonal to fns of form $\phi(B_{t_1},\ldots,B_{t_n})$ where $\phi \in C_0^\infty(\mathbb{R}^n)$
Dense in $L^2(\Omega,\mathcal{F}_T,P) \quad \Rightarrow \quad g = 0$

Lemma

$F \in L^2(\Omega, \mathcal{F}_T, P)$ There exists a unique $f(t, \omega)$ which is

1. progressively measurable
2. square integrable
3. $\mathcal{B}([0,\infty)) \times \mathcal{F}$ measurable

such that

$$F(\omega) = E[F] + \int_0^T f dB.$$

Proof of Uniqueness

suppose

$$F = E[F] + \int_0^T f_1 dB = E[F] + \int_0^T f_2 dB$$

$$\Rightarrow \quad \int_0^T (f_2 - f_1) dB = 0 \quad \Rightarrow \quad \int_0^T E[(f_2 - f_1)^2] dt = 0 \quad \Rightarrow \quad f_2 = f_1$$

Proof of existence

First we prove it if F is of the form $F = e^{\int_0^T hdB - \frac{1}{2}\int_0^T h^2 ds}$
Defining $F_t = e^{\int_0^t hdB - \frac{1}{2}\int_0^t h^2 ds}$ gives

$$dF = hFdB, \qquad F_0 = 1,$$

so

$$F_t = 1 + \int_0^t F_s hdB.$$

Plugging in $t = T$ gives the result.

If F is a linear combination of such functions the result follows by linearity

Proof of existence for $F \in L^2(\Omega, \mathcal{F}_T, P)$

$F_n \in L^2(\Omega, \mathcal{F}_T, P)$ with $F_n \to F$ and

$$F_n = E[F_n] + \int_0^T f_n dB.$$

$E[F_n] \to E[F]$, so wlog $E[F_n] = E[F] = 0$

$$E[(F_n - F_m)^2] = \int_0^T E[(f_n - f_m)^2] dt \to 0 \quad as \quad n, m \to \infty$$

$\Rightarrow \quad f_n$ Cauchy in $L^2([0, T] \times \Omega, dx \times dP)$.

Let f be the limit. Taking limits we have

$$F = E[F] + \int_0^T f dB.$$

Proof of the martingale representation theorem

By previous lemma, for each t we have $\sigma_t(s,\omega)$ such that

$$M_t = E[M_t] + \int_0^t \sigma_t(s) dB_s$$

Let $t_2 > t_1$

$$M_{t_1} = E[M_{t_2} \mid \mathcal{F}_{t_1}]$$

$$\int_0^{t_1} \sigma_{t_2}(s) dB_s = \int_0^{t_1} \sigma_{t_1}(s) dB_s$$

Uniqueness $\Rightarrow \quad \sigma_{t_1} = \sigma_{t_2}$

Quadratic variation of $X_t = \int_0^t \sigma(s) dB_s$

$e^{\lambda \int_0^t \sigma(s) dB_s - \frac{\lambda^2}{2} \int_0^t \sigma^2(s) ds}$ =martingale

$$E[e^{\lambda \int_{t_i}^{t_{i+1}} \sigma(s) dB_s - \frac{\lambda^2}{2} \int_{t_i}^{t_{i+1}} \sigma^2(s) ds} \mid \mathcal{F}_{t_i}] = 0$$

$$E[Z(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}] = 0, \qquad Z(t_i, t_{i+1}) = (\int_{t_i}^{t_{i+1}} \sigma(s) dB_s)^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s) ds$$

$$E[\{Z(t_i, t_{i+1})\}^2 - 4(\int_{t_i}^{t_{i+1}} \sigma^2(s) ds)^2 \mid \mathcal{F}_{t_i}] = 0$$

$$E[(\sum_i Z(t_i, t_{i+1}))^2] \leq 4E[(\sum_i (\int_{t_i}^{t_{i+1}} \sigma^2(s) ds)^2] \to 0$$

$$\langle X_t, X_t \rangle = \int_0^t \sigma^2(s, \omega) ds$$

Levy's Theorem

Let X_t be a process adapted to a filtration $\mathcal{F}_t$ which

1. has continuous sample paths
2. is a martingale
3. has quadratic variation t

Then X_t is a Brownian motion

Proof of Levy's theorem

Enough to show that for each λ,

$$E[e^{i\lambda(X_t - X_s)} \mid \mathcal{F}_s] = e^{-\frac{1}{2}\lambda^2(t-s)}$$

Call $M_t = e^{i\lambda X_t + \frac{1}{2}\lambda^2 t}$, $t_j = s + \frac{j}{2^n}(t-s)$

$$M_t - M_s = \sum_{j=1}^{2^n} M_{t_j} - M_{t_{j-1}}$$

$$= \sum_{j=1}^{2^n} i\lambda M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) - \frac{1}{2}\lambda^2 M_{\xi_j}[(X_{t_j} - X_{t_{j-1}})^2 - (t_j - t_{j-1})]$$

$$E[M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_s] = E[E[M_{t_{j-1}}(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}}] \mid \mathcal{F}_s]$$

$$= E[M_{t_{j-1}} E[(X_{t_j} - X_{t_{j-1}}) \mid \mathcal{F}_{t_{j-1}}] \mid \mathcal{F}_s] = 0$$

Proof of Levy's theorem

Fix m. Let $\xi^m = \max\{\frac{i}{2^m} : \frac{i}{2^m} \le \xi\}$

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} M_{\xi_j^m}[(X_{t_j} - X_{t_{j-1}})^2 - (t_j - t_{j-1})] = 0$$

So we only have to show

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} [M_{\xi_j^m} - M_{\xi_j}](X_{t_j} - X_{t_{j-1}})^2 = 0$$

Would follow from

$$\lim_{n\to\infty} \sum_{j=1}^{2^n} (X_{\xi_j^m} - X_{\xi_j})(X_{t_j} - X_{t_{j-1}})^2 = 0$$

Left hand side $= t \lim_{n\to\infty} \max_{1\le j\le 2^n} |X_{\xi_j^m} - X_{\xi_j}| = 0$ a.s.

Note the same proof gives

Itô formula for semimartingales

Let $M_t^1, \ldots, M_t^d$ be martingales with respect to a filtration $\mathcal{F}_t$, $t \geq 0$, $A_t^1, \ldots, A_t^d$ adapted processes of bounded variation, $X_t = x_0 + A_t + M_t$ where $x_0 \in \mathcal{F}_0$, and $f(t,x) \in C^{1,2}$. Then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) dA_s^i + dM_s^i \\ &\quad + \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d\langle M^i, M^j \rangle_s \end{aligned}$$

Multidimensional Levy's theorem

Let $M_t^1, \ldots, M_t^d$ be continuous martingales with respect to a filtration $\mathcal{F}_t$, $t \geq 0$, with

$$\langle M^i, M^j \rangle_t = \delta_{ij} t$$

Then $M_t^1, \ldots, M_t^d$ is a Brownian motion in $\mathbb{R}^d$

Time change

Let Y_t be a stochastic integral

$$Y_t = \int_0^t g ds + \int_0^t f dB$$

where f and g are adapted square integrable processes
Let $c_t > 0$ be another adapted process and define

$$\beta_t = \int_0^t c_s ds.$$

Then β_t is adapted and strictly increasing. We call α_t its inverse. We can check that

$$Y_{\alpha_t} = \int_0^t \frac{f}{c} ds + \int_0^t \frac{g}{\sqrt{c}} d\tilde{B}$$

for some Brownian motion $\tilde{B}$. In particular, if we are given a stochastic integral $\int_0^t f dB$ we can choose $f^2 = c$ as the rate of our time change and the resulting Y_{α_t} is a Brownian motion

Time change

Theorem

Let B_t be Brownian motion and $\mathcal{F}_t$ its canonical σ-field

Suppose that M_t is a square integrable martingale with respect to $\mathcal{F}_t$

Let

$$M_t = M_0 + \int_0^t f(s) dB_s$$

be its representation in terms of Brownian motion. Suppose that $f^2 > 0$ (i.e. its quadratic variation is strictly increasing)

Let $c = f^2$ and define α_t as above

Then M_{α_t} is a Brownian motion

Example. Stochastic growth model

$$dX = rXdt + \sigma\sqrt{X}dB$$

Solution is $X_t = r\tau_t + B(\tau_t)$ where $\tau_t' = X_t$
Because if

$$dY = rdt + \sigma dB$$

then by time change

$$X_t = Y_{\tau_t}$$

satisfies

$$dX = r\tau' dt + \sigma\sqrt{\tau'}dB$$

Here's a funny trick to solve

$$dX = \sigma\sqrt{X}dB$$

Let $\phi = \phi(t)$ be deterministic and look at

$$Y(t) = e^{-X(t)\phi(T-t)}$$

$$dY = (\dot{\phi} + \frac{\sigma^2}{2}\phi^2)Ydt - \phi Y\sigma\sqrt{X}dB$$

So if $\dot{\phi} = -\frac{\sigma^2}{2}\phi^2$, $\phi(0) = \lambda$ then

$$E[e^{-\lambda X(T)}] = e^{-X(0)\phi(T)}$$

Called Duality
Great if it works.

Example. Cox-Ingersol-Ross model

The interest rate $r(t)$ is assumed to satisfy the equation

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t).$$

Note that the Lipschitz condition is not satisfied, but existence/uniqueness holds by the stronger theorem we did not prove

If $d = 4\alpha/\sigma^2$ is a positive integer, then we can find a solution as follows: Let $B_1(t), B_2(t), \ldots, B_d(t)$ be d independent Brownian motions

and let $X_1(t), X_2(t), \ldots, X_d(t)$ be the solutions of the Langevin equations

$$dX_i = -\alpha X_i dt + \sigma dB_i, \qquad i = 1, \ldots, d.$$

In other words, $X_1(t), X_2(t), \ldots, X_d(t)$ are d independent Ornstein-Uhlenbeck processes

Example. Cox-Ingersol-Ross model

$r(t) = X_1^2(t) + \cdots + X_d^2(t), \quad X_1(t), X_2(t), \ldots, X_d(t)$ indep O-U.

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}\left\{\sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)\right\}.$$

$$d\tilde{B}(t) = \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)$$

$\tilde{B}(t)$ is a martingale with quadratic variation

$$d\tilde{B}d\tilde{B} = \sum_{i,j=1}^{d} \frac{X_i(t)X_j(t)}{r(t)} dB_i(t)dB_j(t) = \sum_{i=1}^{d} \frac{X_i^2(t)}{r(t)} dt = dt$$

$\Rightarrow \tilde{B}(t)$ Brownian motion

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}d\tilde{B}(t)$$

The type of solutions dealt with in the existence/uniqueness theorem are *strong solutions*

This means that you are given the Brownian motion $B(t)$ and asked to come up with a solution $X(t)$ of the stochastic differential equation $dX = bdt + \sigma dB$.

A *weak solution* is when you just find some Brownian motion $\tilde{B}(t)$ for which you can solve the equation $dX = bdt + \sigma d\tilde{B}$.

Example. Cox-Ingersol-Ross.

If $d = 1$ then $\tilde{B}(t) = B_1(t)$ so we have a strong solution

But if $d = 2, 3, \ldots$ then all we have is a weak solution, because given a Brownian motion $B(t)$ it is not at all clear how to find $B_1(t), \ldots, B_d(t)$ for which $dB(t) = \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)$.

Diffusion process as probability measure on $C([0, \infty))$

Brownian motion=collection of rv's B_t, $t \geq 0$ on $(\Omega, \mathcal{F}, P)$

or

Brownian motion=probability measure P on $C([0, \infty))$

If $\mathcal{A}$ is a (measurable) subset of continuous functions, then $P(\mathcal{A})$ is just the probability that a Brownian path falls in that subset

Same for any diffusion.If $X(t)$ is the solution of the stochastic differential equation $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$, $X(0) = x$ then we can let $P_x^{a,b}$ denote the probability measure on the space of continuous functions with

$$P_x^{a,b}(\mathcal{A}) = Prob(X(\cdot) \in \mathcal{A}) \qquad a = \sigma\sigma^T$$

Question:What is the relation of $P_x^{a,b}$ for different x, a, b?

1. If $x_1 \neq x_2$ then $P^{a,b}_{x_1} \perp P^{a,b}_{x_2}$.
2. The quadratic variation

$$\lim_{n\to\infty} \sum_{i=0}^{\lfloor 2^n T \rfloor} |X(\frac{i+1}{2^n}) - X(\frac{i}{2^n})|^2 = \int_0^T a(t, X(t))dt \qquad \text{a.s. } P^{a,b}_x$$

   Hence if $a_1 \neq a_2$, $P^{a_1,b}_x \perp P^{a_2,b}_x$

3. To see what happens if we change b, let $dX_i(t) = b_i dt + \sigma dB(t)$, $i = 1, 2$

$$\begin{aligned} &\frac{P(X_1(t_1) \in dx_1, \ldots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \ldots, X_2(t_n) \in dx_n)} \\ &= e^{-\sum_{i=0}^{n-1} \frac{(x_{i+1}-x_i-b_1(t_{i+1}-t_i))^2-(x_{i+1}-x_i-b_2(t_{i+1}-t_i))^2}{2\sigma^2(t_{i+1}-t_i)}} dx_1 \cdots dx_n \\ &= e^Z dx_1 \cdots dx_n \end{aligned}$$

$$\frac{P(X_1(t_1) \in dx_1, \ldots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \ldots, X_2(t_n) \in dx_n)} = e^Z dx_1 \cdots dx_n$$

$$\begin{aligned}
Z &= -\sum_{i=0}^{n-1} \frac{(x_{i+1} - x_i - b_1(t_{i+1} - t_i))^2 - (x_{i+1} - x_i - b_2(t_{i+1} - t_i))^2}{2\sigma^2(t_{i+1} - t_i)} \\
&= -\sum_{i=0}^{n-1} \sigma^{-1}(b_2 - b_1)\sigma^{-1}(x_{i+1} - x_i - b_2(t_{i+1} - t_i)) \\
&\quad -\frac{1}{2}\sum_{i=0}^{n-1}(\sigma^{-1}(b_2 - b_1))^2(t_{i+1} - t_i) \\
&\overset{n\to\infty}{\longrightarrow} -\int_0^t \sigma^{-1}(b_2 - b_1)dB(s) - \frac{1}{2}\int_0^t(\sigma^{-1}(b_2 - b_1))^2 ds
\end{aligned}$$

Cameron-Martin-Girsanov formula

For each x the measure $P_x^{a,b}$ is absolutely continuous on $\mathcal{F}_t$ with respect to the measure $P_x^{a,0}$ and

$$\left.\frac{dP_x^{a,b}}{dP_x^{a,0}}\right|_{\mathcal{F}_t} = \exp\left\{\int_0^t a^{-1}(X_s)b(X_s)dX_s - \frac{1}{2}\int_0^t b(X_s)a^{-1}(X_s)b(X_s)ds\right\}$$

Proof

We want to show is if we define a measure

$$Q(A) = \int_A \exp\left\{\int_0^t a^{-1}bdX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds\right\} dP_{x_0}^{a,0}$$

then Q is a diffusion with parameters a and b in other words, for each λ,

$$\exp\{\lambda(X_t - X_0 - \int bds) - \frac{\lambda^2}{2}\int_0^t ads\}$$

is a martingale with respect to Q

Proof.

$$Y_t = \int_0^t (\lambda + a^{-1}b)dX_s = \int_0^t (\lambda + a^{-1}b)\sigma dB_s.$$

$$e^{Y_t - Y_0 - \frac{1}{2}\int_0^t a(\lambda + a^{-1}b)^2 ds} = \text{martingale w.r.t. } P_{x_0}^{a,0}$$

$$e^{\lambda(X_t - X_0 - \int_0^t bds) - \frac{\lambda^2}{2}\int_0^t ads + \int_0^t a^{-1}bdX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds} = \text{martingale w.r.t. } P_{x_0}^{a,0}$$

$$e^{\lambda(X_t - X_0 - \int_0^t bds) - \frac{\lambda^2}{2}\int_0^t ads} = \text{martingale w.r.t. } Q$$

$$dQ = e^{\int_0^t a^{-1}bdX_s - \frac{1}{2}\int_0^t a^{-1}b^2 ds} dP_{x_0}^{a,0}$$

□

"Solution" of one dimensional stochastic differential equations

Suppose you want to solve the one dimensional stochastic differential equation

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$$

By Cameron-Martin-Girsanov, you could instead solve

$$dZ(t) = \sigma(t, Z(t))dB(t)$$

and then change to an equivalent measure which will correspond to the solution of $X(t)$

Define $t(\tau)$ by

$$\tau(t) = \int_0^t \sigma^2(u, Y(u))du$$

$$B(\tau) = Z(t(\tau))$$

is a Brownian motion and $Z(t) = B(\tau(t))$.

Only works in $d = 1$

Brownian motion as the limit of random walks

$X_1, X_2, \ldots$ iid Bernoulli $P(X_i = 1) = P(X_i = -1) = 1/2$

$$S_n = X_1 + \cdots + X_n$$

$B_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor tn \rfloor}$ Takes steps $\pm \frac{1}{\sqrt{n}}$ at times $\frac{1}{n}, \frac{2}{n}, \ldots$

Or $\bar{B}_n(t) =$ polygonalized version. Almost the same but continuous

$$B_n(t) \xrightarrow{n \to \infty} \text{Brownian motion } B(t)$$

What does it mean for stochastic processes to converge?

$$\text{dist}(B_n(t_1), \ldots, B_n(t_k)) \to \text{dist}(B(t_1), \ldots, B(t_k)) \qquad k = 1, 2, 3, \ldots$$

Convergenence of finite dimesional distributions
Immediate from (multidimensional) central limit theorem
Same for $\bar{B}_n(t)$

$P_n =$ measure on $C[0, T]$ corresponding to $\bar{B}_n(t)$, $0 \leq t \leq T$

Invariance principle (Donsker's Theorem)

$$P_n \Rightarrow P$$

Much stronger than convergence of finite dimensional distributions

Examples

1. $$dist(\max_{0 \leq m \leq n} \frac{1}{\sqrt{n}} S_m) \to dist(\sup_{0 \leq t \leq 1} B(t))$$

2. $$dist(n^{-1-\frac{k}{2}} \sum_{m=1}^{n} S_m^k) \to dist(\int_0^1 B^k(t)dt)$$

Brownian motion with variance σ^2 and drift b as the limit of random walks

$X_n(t)$ jumps $\frac{1}{\sqrt{n}}\sigma + \frac{1}{n}b$ or $-\frac{1}{\sqrt{n}}\sigma + \frac{1}{n}b$ with probabilities $1/2$ at times $\frac{1}{n}, \frac{2}{n}, \ldots$

$$X_n(t) - b\frac{\lfloor nt \rfloor}{n} = \sigma B_n(t)$$

$$X_n(t) \to \sigma B(t) + bt$$

General local diffusivity $\sigma^2(t,x)$ and drift $b(t,x)$

$X_n(t)$ jumps

$$\frac{1}{\sqrt{n}}\sigma(\tfrac{i}{n}, X_n(\tfrac{i}{n})) + \frac{1}{n}b(\tfrac{i}{n}, X_n(\tfrac{i}{n})) \quad \text{or} \quad -\frac{1}{\sqrt{n}}\sigma(\tfrac{i}{n}, X_n(\tfrac{i}{n})) + \frac{1}{n}b(\tfrac{i}{n}, X_n(\tfrac{i}{n}))$$

with probabilities $1/2$ at times $\frac{i}{n}$, $i = 1, 2, \ldots$ $\quad X_n(t) \to X(t)$

$$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$$

But how to prove it?

Here's another proof that random walks converge to Brownian motions, which does generalize

Recall $B_n(t) = \frac{1}{\sqrt{n}} S_{\lfloor tn \rfloor}$ where $S_n = X_1 + \cdots + X_n$

Let $f \in C^2$

$$f(B_n(t)) = \frac{1}{n} \sum_{i=0}^{\lfloor tn \rfloor - 1} L_n f(B_n(\frac{i}{n}) = \text{ Martingale}$$

$$L_n f(x) = \frac{1}{2} n(f(x + n^{-1/2}) - 2f(x) + f(x - n^{-1/2}))$$

$$L_n f(x) \to \frac{1}{2} f''(x)$$

$$\frac{1}{n} \sum_{i=0}^{\lfloor tn \rfloor - 1} L_n f(B_n(\frac{i}{n}) \to \frac{1}{2} \int_0^t f''(B(s)) ds$$

$$f(B(t)) - \frac{1}{2} \int_0^t f''(B(s)) ds = \text{ martingale } \quad \Rightarrow \quad B(t) \text{ Brownian motion}$$

Really one needs to show that P_n are precompact as a set of probability measures. It is similar to the proof that Brownian motion is continuous, but you just use the martingale formulation directly. The details are long, but the final result is

Theorem

Suppose that

1. $n \int_{|y-x|\leq 1} (y_i - x_i)(y_j - x_j) p_{1/n}(x, dy) \to a_{ij}(x)$ uniformly on compact sets
2. $n \int_{|y-x|\leq 1} (y_i - x_i) p_{1/n}(x, dy) \to b_i(x)$ uniformly on compact sets
3. $np_{1/n}(x, B(x, \epsilon)^C) \to 0$ uniformly on compact sets, for each $\epsilon > 0$

where $a(x)$ and $b(x)$ are continuous. Suppose that we have weak uniqueness for the stochastic differential equation

$$dX = \sigma(X)dB + b(X)dt$$

and let P denote the measure on $C[0, T]$ corresponding to $X(t)$, $0 \leq t \leq T$. Then $P_n \Rightarrow P$