Stochastic Calculus

Lecture 1 : Brownian motion

Stochastic processes

A *stochastic process* is an indexed set of random variables

$$X_t, \quad t \in T$$

i.e. measurable maps from a probability space $(\Omega, \mathcal{F}, P)$ to a state space $(E, \mathcal{E})$ T = time

In this course $T = \mathbb{R}_+$ or $\mathbb{R}$ (continuous time)

But you could have $T = \mathbb{N}_+$ or $\mathbb{N}$ (discrete time), or other things

In this course $E = \mathbb{R}$ or $\mathbb{R}^d$ $\qquad \mathcal{E} = \mathcal{B}(\mathbb{R}^d)$= Borel σ-field
= smallest σ-field containing open sets

Fix $\omega \in \Omega$. $X_t(\omega)$ is a function of $t \in \mathbb{R}_+$.
So X_t is a random function.
In this course we will be studying random *continuous* functions

Real random variables

A *probability space* $(\Omega, \mathcal{F}, P)$ consists of a set Ω, on which there is a σ*-field* $\mathcal{F}$

- if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
- if $A_i \in \mathcal{F}$, $i = 1, 2, 3, \ldots$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$

and a *probability measure*

- $0 = P(\emptyset) \leq P(A) \leq P(\Omega) = 1$, $A \in \mathcal{F}$
- if $A_i \in \mathcal{F}$, $i = 1, 2, 3, \ldots$ are disjoint, $A_i \cap A_j = \emptyset$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

A (real-valued) *random variable* is a measurable map

$$X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

Distribution function $F(x) = P(X \leq x)$

Measure $\mu(A) = P(X \in A)$

Characteristic function $\varphi(t) = E[e^{itX}] = \int e^{itx} dF(x)$

Random variables in $\mathbb{R}^n$

$$n \text{ random variables } X_1, \dots, X_n$$
$$= \text{random } n\text{-vector } X = (X_1, \dots, X_n)$$
$$= \text{random element of } \mathbb{R}^n$$

Distribution function

$$F(x_1, \dots, x_n) = P(X_1 \leq x_1, \dots, X_n \leq x_n)$$

Measure $\mu(A) = P(X \in A)$

Characteristic function $\varphi(t_1, \dots, t_n) = E[e^{it \cdot X}]$.

Distribution of a stochastic process

Analogously: Discrete time stochastic process $X_1, X_2, \ldots$ is a random sequence or a random element of $\prod_{i=1}^{\infty} \mathbb{R}$

Continuous time stochastic process $X_t,\ t \geq 0$ is a random function or a random element of $\Omega = \prod_{t \geq 0} \mathbb{R}$

Family of finite dim. distr.'s, one for each $t_1 < \cdots < t_n$,

$$F_{t_1,\ldots,t_n}(x_1, \ldots, x_n) = P(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n) \tag{1}$$

Consistency: $F_{t_1,\ldots,\hat{t}_i,\ldots,t_n}(x_1, \ldots, \hat{x}_i, \ldots, x_n) = F_{t_1,\ldots,t_n}(x_1, \ldots, \infty, \ldots, x_n)$

$\mathcal{F}$ generated by *cylinder sets* $E = \{X_{t_1}, \ldots, X_{t_n} \in A\}$, $A \in \mathcal{B}(\mathbb{R}^n)$

Kolmogorov Extension Theorem

Let $\{F_{t_1,\ldots,t_n}\}_{t_1,\ldots,t_n \in \mathbb{R}^n, n=1,2,\ldots}$ *be a consistent family of finite dimensional distributions. There is a unique probability measure* P *on* $(\Omega, \mathcal{F})$ *satisfying (1).*

Gaussian processes

$X \sim \mathcal{N}(m, \sigma^2)$ if X is Gaussian with mean m and variance σ^2

$F(x) = \int_{-\infty}^{x} f(y)dy$, density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$ $\varphi(t) = e^{itm - \frac{1}{2}\sigma^2 t^2}$

$X = (X_1, \ldots, X_n) \sim \mathcal{N}(m, C)$ if it has n-dimensional density

$$\frac{1}{\sqrt{2\pi \det C}} e^{-\frac{1}{2}(x-m)^T C^{-1}(x-m)}$$

A process is Gaussian if all finite dimensional distributions are Gaussian

Proposition

If (X_1, X_2) is Gaussian, then they are independent if and only if they are orthogonal, i.e. $C_{12} = C_{21} = E[(X_1 - m_1)(X_2 - m_2)] = 0$.

Brownian motion

Brownian motion B_t, $t \geq 0$ is a Gaussian process with independent, stationary increments $B_{t+h} - B_t$

Independent increments: $B_{t_4} - B_{t_3}$, $B_{t_2} - B_{t_1}$ independent if $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$

Stationary: $B_{t+h} - B_t \stackrel{dist}{=} B_{s+h} - B_s$

$B_{t+h} - B_t \sim \mathcal{N}(0, h)$

$$P(B_{t_1} \leq x_1, \ldots, B_{t_n} \leq x_n) =$$
$$\int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \frac{e^{-\frac{y_1^2}{2t_1}}}{\sqrt{2\pi t_1}} \frac{e^{-\frac{(y_2-y_1)^2}{2(t_2-t_1)}}}{\sqrt{2\pi(t_2-t_1)}} \cdots \frac{e^{-\frac{(y_n-y_{n-1})^2}{2(t_n-t_{n-1})}}}{\sqrt{2\pi(t_n-t_{n-1})}} dy_1 \cdots dy_n$$

Basic properties of Brownian motion

Mean: $m_t = E[B_t] = 0$

Covariance: If $s < t$,
$E[B_t B_s] = E[(B_t - B_s)B_s] + E[B_s^2] = E[B_t - B_s]E[B_s] + s = s$

B_t is Gaussian process with

$$m_t = 0 \quad C_{s,t} = E[B_t B_s] = \min(s,t)$$

Proposition

Let B_t, $t \geq 0$ be a Brownian motion.

1. *For any $s \geq 0$, $\tilde{B}_t = B_{t+s} - B_s$, $t \geq 0$ is a Brownian motion independent of B_u, $u \leq s$*
2. $-B_t$, $t \geq 0$ *is a Brownian motion*
3. *For any a, $aB_{a^{-2}t}$, $t \geq 0$ is a Brownian motion*
4. $tB_{1/t}$, $t \geq 0$ *is a Brownian motion.*

Continuity of Brownian motion

We will construct Brownian motion directly on $\Omega = C([0,1])$,with its Borel σ-field, under the sup norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.

Lemma

If X is $\mathcal{N}(0,1)$ then $P(|X| \geq a) \leq e^{-a^2/2}$.

Proof.

We use the exponential Tchebyshev's inequality: For $\lambda > 0$,

$$P(|X| \geq a) \leq E[e^{\lambda X}]/e^{\lambda a} = e^{-\lambda a + \lambda^2/2}$$

Choose $\lambda = a$. □

Continuity of Brownian motion: Haar functions

Definition

Haar basis of $L^2[0,1]$

$$\mathcal{H}_{n,j}(x) = 2^{\frac{n+1}{2}} \left\{ \mathbf{1}_{[\frac{2j}{2^{n+1}}, \frac{2j+1}{2^{n+1}})} - \mathbf{1}_{[\frac{2j+1}{2^{n+1}}, \frac{2j+2}{2^{n+1}})} \right\}$$

Integrals are Schauder functions $\mathcal{S}_{n,j}(x) = \int_0^x \mathcal{H}_{n,j}(y)dy$
Tent of height $2^{-\frac{n+1}{2}}$ *between* $\frac{2j}{2^{n+1}}$ *and* $\frac{2j+2}{2^{n+1}}$.

Definition

$B_n(t) =$ *polygonal approx. on the points* $0, 1/2^n, 2/2^n, \ldots, 1$
$B_{n+1}(\frac{2j+1}{2^{m+1}}) - B_n(\frac{2j+1}{2^{m+1}}) = B_{n+1}(\frac{2j+1}{2^{m+1}}) - \frac{B_{n+1}(\frac{2j+2}{2^{m+1}}) + B_{n+1}(\frac{2j}{2^{m+1}})}{2}$

$$B_{n+1}(t) - B_n(t) = \sum_{j=0}^{2^n-1} \xi_{j,n}\mathcal{S}_{j,n}(x) \qquad \xi_{j,m} \text{ indep } \mathcal{N}(0,1)$$

Continuity of Brownian motion

Theorem

Brownian motion has continuous sample paths

Proof.

Polygonal approx. $B_n(t,\omega)$ *is continuous for each* ω

Enough to show: $B_n(\omega)$ *converges uniformly a.s.*

i.e. $B(t) = \sum_{m=0}^{\infty} B_{m+1}(t) - B_m(t) + B_0(t)$ *converges uniformly*

$B_n(t) - B(t) = \sum_{m=n}^{\infty} B_{m+1}(t) - B_m(t)$

$\sup_{0\leq t\leq 1} |B_n(t) - B(t)| \leq \sum_{m=n}^{\infty} \sup_{0\leq t\leq 1} |B_{m+1}(t) - B_m(t)|$

Need: $\forall \epsilon > 0$, $\exists N(\omega) < \infty$ *a.s., such that for all* $n \geq N(\omega)$,

$$\sum_{m=n}^{\infty} \sup_{0\leq t\leq 1} |B_{m+1}(t) - B_m(t)| \leq \epsilon.$$

Continuity of Brownian motion: Proof continued

Proof.

$$\sup_{0 \le t \le 1} |B_{m+1}(t) - B_m(t)| = 2^{-\frac{m+1}{2}} \max_{0 \le j \le 2^m - 1} |\xi_{j,m}|.$$

$$\begin{aligned} P(2^{-\frac{m+1}{2}} \max_{0 \le j \le 2^m-1} |\xi_{j,m}| \ge 2^{-\frac{m}{4}}) &\le 2^m P(2^{-\frac{m+1}{2}} |\xi_{0,m}| \ge 2^{-\frac{m}{4}}) \\ &\le 2^m e^{-2^{m/2}} \end{aligned}$$

$$\sum_{n=1}^{\infty} P(\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0 \le j \le 2^m-1} |\xi_{j,m}| \ge \sum_{m=n}^{\infty} 2^{-m/4})$$

$$\le \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} 2^m e^{-2^{m/2}} < \infty$$

Continuity of Brownian motion: Proof continued

Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} P(A_n) < \infty$ *then almost every* ω *is in at most finitely many* A_n*.*

Proof.

It follows that $\exists N_1(\omega)$ s.t. for $n \geq N_1(\omega)$

$$\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0 \leq j \leq 2^m - 1} |\xi_{j,m}| \leq \sum_{m=n}^{\infty} 2^{-m/4}$$

$\forall \epsilon$, r.h.s.$\leq \epsilon$ for all $n \geq N_2$. □

Hölder continuity

f is locally Hölder of order α if for every $L < \infty$

$$\sup_{0 \leq s < t \leq L} \frac{|f(t) - f(s)|}{|t - s|^{\alpha}} < \infty$$

Theorem

Let X_t, $t \geq 0$ be a stochastic process for which $\exists \gamma, C, \delta > 0$,

$$E[|X_t - X_s|^{\gamma}] \leq C|t - s|^{1+\delta}$$

Then X_t is a.s. locally Hölder continuous of order $\alpha < \delta/\gamma$

Example: Brownian motion is Hölder $\alpha < 1/2$

$$E[|B_t - B_s|^{2p}] = \int x^{2p} \frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dx = C_p|t - s|^p$$