Problem Set 5: Due Dec 9

1. (Brownian motion as a Fourier series.) Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Show that for each $t > 0$,

$$B(t) = \sum_{n=1}^{\infty} X_n n^{-1} \sin(\pi n t)$$

   converges. In what sense?

2. Let $X_1, X_2, \ldots$ be i.i.d and nondegenerate. Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} X_n z^n$$

   is 0 or 1 a.s. according to whether $E[\max(\log|X_1|, 0)]$ is finite or not. (The radius of convergence is $\sup\{a \;:\; \sum_{n=1}^{\infty} |X_n| a^n < \infty\}$)

3. Let $X_1, X_2, \ldots$ be i.i.d. and $S_n = X_1 + \cdots + X_n$. Show that if $S_n/n \to 0$ in probability then $\max_{1 \le i \le n} S_i/n \to 0$ in probability.

4. Let N have a Poisson distribution with mean λ and let $X_1, X_2, \ldots$ be independent with $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$. Let N' be the number of $m \le N$ with $X_m = 1$. Show that N' is a Poisson distribution with mean λp.

5. Let X have a symmetric stable distribution with index α. Show that $E[|X|^p] < \infty$ for $p < \alpha$.

6. Let X_1 and X_2 be independent standard normals. Find the distribution of X_1/X_2.

7. Show that the gamma distribution is infinitely divisible.

8. Show that the characteristic function φ of an infinitely divisible distribution never vanishes. (Hint: First show that $|\varphi|^2$ is also infinitely divisible.)

9. Suppose that $\mathbf{X} = (X_1, \dots, X_d)$ has a multivariate normal distribution with mean vector $\mathbf{m}$ and covariance matrix $\mathbf{C}$. Show that $X_1, \dots, X_d$ are independent if and only if $\mathbf{C}$ is a diagonal matrix.

10. Show that $\mathbf{X} = (X_1, \dots, X_d)$ has a multivariate normal distribution with mean vector $\mathbf{m}$ and covariance matrix $\mathbf{C}$ if and only if for each vector $\mathbf{v} = (v_1, \dots, v_d)$, $\mathbf{v} \cdot \mathbf{X}$ is normal with mean $\mathbf{v} \cdot \mathbf{m}$ and variance $\mathbf{v} \cdot \mathbf{C}\mathbf{v}$.