Problem Set 4: Due Nov 18

1. Find an example of random variables X_n with densities f_n whose distributions converge to the uniform distribution on $(0,1)$ but whose densities do not converge to $1_{(0,1)}$ pointwise.

2. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random vectors (i.e. with values in $\mathbf{R}^d$) with mean $m \in \mathbf{R}^d$ and covariance C, a $d \times d$ matrix. Find the limit in distribution of

$$\sqrt{n}\left[\frac{X_1 + \cdots + X_n}{n} - m\right].$$

3. Suppose that $X_1, X_2, \ldots$ are independent with mean zero and $\mathrm{Var}(X_j) = \sigma_j^2$. Define $s_n^2 = \sum_{j=1}^n \sigma_j^2$ and assume $s_n^2 \to \infty$ as $n \to \infty$.

   (a) (Lindeberg's condition) Suppose that for each $\epsilon > 0$,

   $$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^n E[X_j^2 1_{\{|X_j| \geq \epsilon s_n\}}] = 0.$$

   Prove the central limit theorem for $s_n^{-1}[X_1 + \cdots + X_n]$.

   (b) Lyapunov's condition is that there exists $\delta > 0$ such that

   $$\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^n E[X_j^{2+\delta}] = 0.$$

   Prove that this implies Lindeberg's condition.

   (c) In the special case of independent random variables X_j with $P(X_j = a_j) = P(X_j = -a_j) = 1/2$ what do Lindeberg's and Lyapunov's conditions demand of the sequence a_j? Are there sequences so that Lindeberg's condition holds, but not Lyapunov's? Are there sequences so that the central limit theorem does not hold?

4. For each n, let X_n be an $\mathbf{R}^n$ valued random vector, with uniform distribution on the sphere of radius $\sqrt{n}$. Show that the distribution of the first coordinate X_n^1 where $X_n = (X_n^1, \ldots, X_n^n)$ converges to a normal. (Hint. Write $X_n^i = Z_n^{-1} Y_n^i$ where Y_n^i are i.i.d. normals and $Z_n^2 \sim \sum_{i=1}^n |Y_n^i|^2$.)

5. Show that
$$d(X,Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right]$$
is a metric on the set of random variables on $(\Omega, \mathcal{F}, P)$ and that $d(X_n, X) \to 0$ if and only if $X_n \to X$ in probability.

6. Let $X_1, X_2, \ldots$ be i.i.d. with characteristic function φ.
   (a) Show that if $\varphi'(0) = ia$ then $\frac{1}{n}\sum_{j=1}^n X_j \to a$ in probability.
   (b) Show that if $\frac{1}{n}\sum_{j=1}^n X_j \to a$ in probability, then $\varphi(\frac{t}{n})^n \to e^{iat}$. Conclude that $\varphi'(0) = ia$.
   (c) Prove that the weak law of large numbers holds if and only if $\varphi'(0)$ exists.

7. Let $X_1, X_2, \ldots$ be i.i.d. with mean zero and finite variance. Prove the central limit theorem for
$$\frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^2}}.$$

8. Let $X_1, X_2, \ldots$ be i.i.d. with distribution function F. The empirical distribution function is
$$F_n(x) = \frac{1}{n}\sum_{j=1}^n 1_{\{X_j \le x\}}.$$
   (a) Why is it called that?
   (b) Find the limit of $F_n(x)$
   (c) Find the limiting distribution of $G_n(x) = \sqrt{n}(F_n(x) - E[F_n(x)])$
   (d) Find the limiting distribution of the random $d-$vector $(G_n(x_1), \ldots, G_n(x_d))$.

9. Show that if $\int |x|^n dF < \infty$ then the characteristic function φ of distribution F has a continuous nth derivative given by $\varphi^{(n)}(t) = \int (ix)^n e^{itx} dF$ and use this to show that for a standard normal random variable

$$E[X^{2n}] = \frac{(2n)!}{2^n n!}$$