Problem Set 3: Due Oct 28

1. Give an example to show that the weak law of large numbers can hold even if the expectation is not finite. (Any example is fine, but here is one possibility: Take $X_1, X_2, \ldots$ i.i.d. with $P(X_1 = (-1)^n n) = Cn^{-2}(\log n)^{-1}$.)

2. Let $X_1, X_2, \ldots$ be independent random variables with Poisson distribution with $E[X_n] = m_n$. Show that if $\sum_{n=1}^{\infty} m_n = \infty$ then

$$\frac{X_1 + \cdots + X_n}{m_1 + \cdots + m_n} \to 1$$

almost surely.

3. If $X_1, X_2, \ldots$ are random variables, then there is a sequence of real numbers $a_1, a_2, \ldots$ so that

$$\frac{X_n}{a_n} \to 0$$

almost surely.

4. Let $Z_1, Z_2, \ldots$ be i.i.d. Normals, mean 0 and variance 1. How big is

$$M_n = \max\{Z_1, \ldots, Z_n\} \quad ?$$

5. Let $X_1, X_2, \ldots$ be i.i.d. with exponential distribution with mean 1 and $M_n = \max\{X_1, \ldots, X_n\}$. Show that, suitably normalized, M_n converges weakly to a Gumbel. (The Gumbel distribution is $e^{-e^{-y}}$.)

6. (A local limit theorem) Let $X_1, X_2, \ldots$ be independent Poisson random variables with mean 1. Compute $p_{n,k} = P(S_n = k)$ explicitly and use Stirling's formula,

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}$$

to show that the $p_{n,k}$, suitably normalized, converge to the normal density. (Part of the problem here is to figure out what the normalization should be.)

7. Show that

$$d_{\text{Levy}}(F,G) = \inf\{\epsilon > 0 : F(x-\epsilon)-\epsilon \leq G(x) \leq F(x+\epsilon)+\epsilon \quad \text{for all} \quad x\}$$

is a metric on the space of distribution functions. Show that convergence in this metric is equivalent to weak convergence.

8. Prove or disprove the following.

   (a) If $F_n \Rightarrow F$ and F is continuous, then $F_n \to F$ uniformly.

   (b) If $X_n \stackrel{\text{prob}}{\to} X$ then $X_n \Rightarrow X$ (recall this means $F_n \Rightarrow F$ where F_n, F are the d.f.'s of X_n, X).

   (c) If $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$ then $X_n + Y_n \Rightarrow X + Y$.

9. Show that if φ is a characteristic function, then Re φ and φ^2 are characteristic functions.

10. Show that $e^{-|t|^\alpha}$ is *not* a characteristic function if $\alpha > 2$ (Hint: Check a couple of derivatives.)