Problem Set 2
Due October 14

1. Show (i) if X and Y are independent, then $\sigma(X)$ and $\sigma(Y)$ are independent (ii) if $\mathcal{F}$ and $\mathcal{G}$ are independent and $X \in \mathcal{F}$ and $Y \in \mathcal{G}$, then X and Y are independent.

2. In each case, let $X_1, X_2, \ldots$ be independent and identically distributed with the given distribution. Compute the distribution of $X_1 + \cdots + X_n$.

   (a) n-dimensional Gaussian with mean m and covariance matrix C

   (b) gamma density with parameters 1 and λ

   (c) Poisson distribution with parameter λ

   (d) Binomial distribution with parameters p and n.

3. Let X and Y be non-negative independent random variables with distribution functions F and G. Find the distribution of XY.

4. (i) Suppose that $X \geq 0$ is an integer valued random variable. Show that $E[X] = \sum_{n \geq 1} P(X \geq n)$. (ii) Find a similar expression when X has a continuous distribution function F (you can make any reasonable assumptions you need on F.)

5. (The Monte Carlo method) (i) Let f be a density function on $\mathbf{R}^d$ and let $X_1, X_2, \ldots$ be independent and identically distributed with density f. Suppose that ϕ is a nice function $\mathbf{R}^d \to \mathbf{R}$. ("nice" means a continuous function, as many times differentiable as you like, and going to zero as fast as you like for large $|x|$.) Find the limiting integral $\lim_{n\to\infty} \frac{\phi(X_1)+\cdots+\phi(X_n)}{n}$
   (ii) In what sense does the limit hold?

   (iii) Can you estimate the probability that $\frac{\phi(X_1)+\cdots+\phi(X_n)}{n}$ is more than $Cn^{-1/2}$ from the limiting integral?

   (iv) If you were to try to evaluate the limiting integral by, say, Riemann sum approximation, about how many lattice points would you have to use to get an accuracy of $n^{-1/2}$? Which is better?

6. (i) Let $X_1, X_2, \ldots$ be uncorrelated with $E[X_n] = m_n$ and $\lim_{n\to\infty} \frac{\mathrm{Var}(X_n)}{n} = 0$. Let $S_n = X_1 + \cdots + X_n$ and $k_n = m_1 + \cdots + m_n$. Show that $\frac{S_n - k_n}{n} \to 0$ in probability.

   (ii) Let $X_1, X_2, \ldots$ be a sequence of random variables with $E[X_n] = 0$ and $\mathrm{Cov}(X_n, X_m) \leq f(n-m)$ for $n \geq m$ with $\lim_{n\to infty} f(n) = 0$. Show that the weak law of large numbers holds.

7. Show that if $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}) < \infty$ then $P(A_n \text{ i.o.}) = 0$.