MAT 135Y

Answers to Term-test 1 Nov. 2004

PART A: 1. $\frac{1}{5}$ 2. $\frac{17}{6}$ 3. $\frac{1+3x}{4-2x}$ 4. $\frac{-\sqrt{7}}{3}$ 5. $y=-\frac{5}{4}$

6. -2 7. $-80\sqrt{5}$ ft/sec. 8. $f'(3)$ exists but $f''(3)$ does not exist.

9. $\frac{1}{4}$ 10. $\frac{-1}{2(1+x^2)}$

A8. By using the definition of the derivative (i.e. from first principles), you can easily show that $f'(3)=0$ and $f''(3)$ does not exist.

Note: $f(x)=\begin{cases}(x-3)^2 & \text{if } x\geq 3\\ -(x-3)^2 & \text{if } x<3.\end{cases}$

A9. given limit $= \lim_{x\to 0}\left(\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}\cdot\frac{\sqrt{1+\tan x}+\sqrt{1+\sin x}}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right)$

$$= \lim_{x\to 0}\frac{\tan x-\sin x}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}$$

$$= \lim_{x\to 0}\frac{(\sin x/\cos x)-\sin x}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}$$

$$= \lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$

$$= \lim_{x\to 0}\frac{(\sin x)(1-\cos x)}{x^3(\sqrt{1+\tan x}+\sqrt{1+\sin x})}\cdot\frac{1+\cos x}{1+\cos x}$$

$$= \lim_{x\to 0}\left(\frac{\sin^3 x}{x^3}\right)\cdot\left(\frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}\right)\cdot\left(\frac{1}{1+\cos x}\right)=\frac{1}{4}.$$

A10. $\frac{dy}{dx}=\frac{1}{1+(\sqrt{1+x^2}-x)^2}\cdot\left(\frac{x}{\sqrt{1+x^2}}-1\right)$

$$=\frac{1}{1+(1+x^2-2x\sqrt{1+x^2}+x^2)}\cdot\left(\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$=\frac{x-\sqrt{1+x^2}}{2(1+x^2-x\sqrt{1+x^2})\sqrt{1+x^2}}$$

$$=\frac{x-\sqrt{1+x^2}}{2\{\sqrt{1+x^2}(\sqrt{1+x^2}-x)\}\sqrt{1+x^2}}=\frac{-1}{2(1+x^2)}$$

B1. Trivial!

B2. (a) $(2+x^2)\sec^2 x+2x\tan x$ (b) $\frac{3(2+x^3)e^{3x}-3x^2e^{3x}}{(2+x^3)^2}$

(c) $\sec(1+\sqrt{x})\tan(1+\sqrt{x})\cdot\frac{1}{2\sqrt{x}}$ (d) $3^{\arcsin(2x)}\cdot\ln 3\cdot\frac{2}{\sqrt{1-4x^2}}$

B3. Let the line through $P(0,-32)$ be tangent to the curve $y=2x^3$ at $Q(a, 2a^3)$.

$\frac{dy}{dx} = 6x^2 \Rightarrow \left.\frac{dy}{dx}\right|_Q = 6a^2 = \text{slope of } \overline{PQ} = \frac{2a^3-(-32)}{a-0}$

$\Rightarrow 6a^3 = 2a^3+32 \Rightarrow 4a^3 = 32 \Rightarrow a^3 = 8 \Rightarrow a=2.$

Hence slope of $\overline{PQ} = 6a^2 = 6(2)^2 = 24$. Hence $\overline{PQ}$ has equation $y-(-32) = 24(x-0)$, i.e. $y = 24x-32$.

B4. $3y^2y' + 4yy' + (xy'+y) - 2y' + 1 = 0$.

Hence $\frac{dy}{dx} = \frac{-(1+y)}{3y^2+4y+x-2}$.

When $x=3$, $y^3+2y^2+3y-2y+3=1$.

Hence $y^3+2y^2+y+2=0$. So $y^2(y+2)+(y+2)=0$.

Hence $(y+2)(y^2+1)=0$. Hence $y=-2$.

Hence, when $x=3$, $\frac{dy}{dx} = \frac{-(1-2)}{3(4)+4(-2)+3-2} = \frac{1}{5}$.

NOTE: -3 marks if you don't have $y=-2$ and $\frac{dy}{dx}=\frac{1}{5}$.

B5. (a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+2) = 2$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 1$.

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, hence f is not continuous at 0.

Hence f is not differentiable at 0. Hence $f'(0)$ does not exist.

(b) f continuous at $2 \Leftrightarrow \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = f(2)$

$\Leftrightarrow \lim_{x\to 2^+} \frac{c(x+1)(x-2)}{x-2} = \lim_{x\to 2^-}(cx^2+1) = 4c+1$

$\Leftrightarrow 3c = 4c+1 \Leftrightarrow c = \underline{\underline{-1}}$.

B6. Let $u = x^{1/5}$. Then $x^{1/7} = (x^{1/5})^{5/7} = u^{5/7}$.

Hence $\lim_{x\to 1} \frac{x^{1/7}-1}{x^{1/5}-1} = \lim_{u\to 1} \frac{u^{5/7}-1}{u-1}$

$= \lim_{u\to 1} \frac{f(u)-f(1)}{u-1}$ (by letting $f(u) = u^{5/7}$)

$= f'(1)$ (by the definition of the derivative)

$= \frac{5}{7}(1)^{-2/7}$ (since $f(u) = u^{5/7}$, hence $f'(u) = \frac{5}{7}u^{-2/7}$)

$= \underline{\underline{\frac{5}{7}}}$.

NOTE: This question is marked very strictly. Very little credit (in almost all cases, zero) is awarded unless your solution is completely correct. You will get zero if you use L'Hospital's Rule.