MAT 135Y

Answers to Term-test 1 (Nov., 2002)

PART A: 1. $\frac{5}{4}$ 2. $-\frac{1}{25}$ 3. $\frac{4}{3}$ 4. $-\frac{5}{6}$ 5. $\frac{4x-3}{5x+2}$

6. 11 7. $(0,-8)$ 8. 0 9. 42 10. $3\sqrt{2}$

A.8 $\lim_{x\to 0}(\cot x - \csc x) = \lim_{x\to 0}\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right)$

$= \lim_{x\to 0}\left(\frac{\cos x - 1}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right)\left(\frac{x}{\sin x}\right) = (0)(1) = 0.$

A.9 When $x=1$, $1+y+y^3=1$. Hence $y+y^3=0$.
Hence $y(1+y^2)=0$. Hence $y=0$.
Now, use implicit differentiation 3 times.
One can then easily get: When $x=1$, $y=0$, $y'=-2$, $y''=2$, $y'''=42$.

A.10 Since the curve is symmetrical about the y-axis and there exists <u>only one</u> line with the given property, this line must be a horizontal line! Hence, $\frac{dy}{dx}=0$ at the two points of tangency. $\frac{dy}{dx}=4x^3-18x=2x(2x^2-9)$, which $=0$ when $x=0, \pm 3/\sqrt{2}$. Hence, by symmetry, the two points of tangency must be at $\pm\frac{3}{\sqrt{2}}$. Hence the distance between the two points must be $2\left(\frac{3}{\sqrt{2}}\right)$, i.e. $3\sqrt{2}$.

B.1 $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$

$= \lim_{h\to 0}\frac{(x^3+3x^2h+3xh^2+h^3)-x^3}{h}$

$= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2.$

B.2 (a) $\frac{dy}{dx} = \frac{4+x^2}{\sqrt{1-x^2}} + 2x\arcsin x$

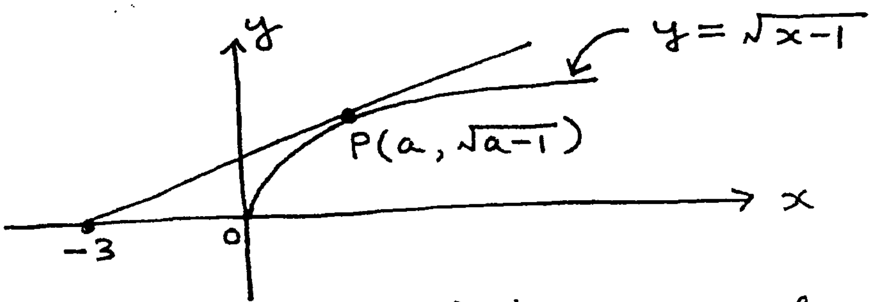
(b) $\frac{dy}{dx} = \frac{(5+e^x)/2\sqrt{x} - \sqrt{x}\,e^x}{(5+e^x)^2}$

(c) $\frac{dy}{dx} = \frac{2x}{1+x^4}$

(d) $\frac{dy}{dx} = 2^{\sin 5x}(\ln 2).5\cos(5x)$

B.3 When $50+96t-16t^2=178$, $16t^2-96t+128=0$.
Hence $16(t-4)(t-2)=0$. Hence $t=2$ (on the way up).
$v=96-32t$. When $t=2$, $v=96-32(2)=\underline{\underline{32}}$ ft./sec.

B.4



Let the point of tangency be $P(a, \sqrt{a-1})$.

$\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$. Slope of tangent at $P = \frac{1}{2\sqrt{a-1}} = \frac{\sqrt{a-1} - 0}{a-(-3)}$.

Hence $2(a-1) = a+3$. Hence $a = 5$.

Hence slope at $P = \frac{1}{2\sqrt{5-1}} = \frac{1}{4}$.

Thus required tangent line has equation $y = \frac{1}{4}(x+3)$.

B.5(a) f continuous at 3 $\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x \to 3^+} f(x) = f(3)$

$\Rightarrow \lim_{x\to 3^-}(2x+c^2) = \lim_{x\to 3^+}(cx+c+2) = 3c+c+2$

$\Rightarrow 6+c^2 = 4c+2 \Rightarrow c^2-4c+4=0 \Rightarrow (c-2)^2=0 \Rightarrow c=2$.

(b) $\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x)-0}{x-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$

$= f'(0)$, which exists by assumption.

Since $\lim_{x\to 0} g(x)$ exists but $g(0)$ is undefined, hence g has a removable discontinuity at 0.

<u>NOTE</u>: No credit will be awarded for this question unless your justification is completely correct.

B.6 $\lim_{x\to 0^-}\frac{\sqrt{-a|x|+b}-\sqrt{3}}{\sqrt{x+4}-2} = \lim_{x\to 0^-}\frac{\sqrt{ax+b}-\sqrt{3}}{\sqrt{x+4}-2}$, which is of the form $\frac{\sqrt{b}-\sqrt{3}}{0}$. So, for the limit to exist, b must be $\underline{\underline{3}}$.

If $b=3$, then limit

$= \lim_{x\to 0^-}\left(\frac{\sqrt{ax+3}-\sqrt{3}}{\sqrt{x+4}-2}\right)\left(\frac{\sqrt{ax+3}+\sqrt{3}}{\sqrt{ax+3}+\sqrt{3}}\right)\left(\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}\right)$

$= \lim_{x\to 0^-}\frac{ax(\sqrt{x+4}+2)}{x(\sqrt{ax+3}+\sqrt{3})} = \lim_{x\to 0^-}\frac{a(\sqrt{x+4}+2)}{\sqrt{ax+3}+\sqrt{3}}$

$= \frac{4a}{2\sqrt{3}} = \frac{2a}{\sqrt{3}} = 4$. Hence $a = \frac{4\sqrt{3}}{2} = \underline{\underline{2\sqrt{3}}}$.

<u>NOTE</u>: This question is marked extremely strictly. Very little or no credit will be given unless your solution is totally correct. Most students got <u>zero</u> (some got 1 or 2 marks).