MAT135Y

Solutions to Term-test 3 (March 2006)

PART A 1. 9 2. −6 3. 128 4. $8/3$ 5. $24/5$ 6. $\pi/6$

7. 22 8. $\frac{1}{5e}$ 9. $\frac{4\pi+3V}{6\pi}$ 10. $\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$ 11. $\frac{23}{2}$ 12. $-4+19\sqrt{2}$

A9. By cylindrical shells, $V=\int_0^1 2\pi(a-x)y\,dx$

$=\int_0^1 2\pi(a-x)(2x)\,dx=4\pi\left(\frac{a}{2}-\frac{1}{3}\right) \Rightarrow \frac{V}{4\pi}=\frac{a}{2}-\frac{1}{3}$

$\Rightarrow \frac{a}{2}=\frac{1}{3}+\frac{V}{4\pi}=\frac{4\pi+3V}{12\pi} \Rightarrow a=\frac{4\pi+3V}{6\pi}$

A10. $\int_{\sqrt{2}-1}^{1}\frac{1}{\sqrt{x^2+2x}}\,dx=\int_{\sqrt{2}-1}^{1}\frac{1}{\sqrt{(x+1)^2-1}}\,dx$

$=\int_{\pi/4}^{\pi/3}\frac{\sec\theta\tan\theta\,d\theta}{\sqrt{\sec^2\theta-1}}\,d\theta$ (by letting $x+1=\sec\theta$)

$=\int_{\pi/4}^{\pi/3}\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|\Big]_{\pi/4}^{\pi/3}=\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right).$

A11. $x<2 \Rightarrow f(x)=|x-2|=2-x \Rightarrow |f(x)-3|=|(2-x)-3|=|1+x|.$

$x\geq 2 \Rightarrow f(x)=|x-2|=x-2 \Rightarrow |f(x)-3|=|(x-2)-3|=|x-5|.$

Hence $\int_{-3}^{6}|f(x)-3|\,dx=\int_{-3}^{2}|1+x|\,dx+\int_2^6|x-5|\,dx$

$=\int_{-3}^{-1}|1+x|\,dx+\int_{-1}^{2}|1+x|\,dx+\int_2^5|x-5|\,dx+\int_5^6|x-5|\,dx$

$=\int_{-3}^{-1}(-1-x)\,dx+\int_{-1}^{2}(1+x)\,dx+\int_2^5(-x+5)\,dx+\int_5^6(x-5)\,dx=\frac{23}{2}.$

A12. $G'(x)=2\int_{4x-5}^{2-2x}\sqrt{1+u^4}\,du-3\int_{6x-5}^{2-3x}\sqrt{1+u^4}\,du.$

$G''(x)=2\left\{-2\sqrt{1+(2-2x)^4}-4\sqrt{1+(4x-5)^4}\right\}$

$\quad -3\left\{-3\sqrt{1+(2-3x)^4}-6\sqrt{1+(6x-5)^4}\right\}.$

Hence $G''(1)=2\{-2\sqrt{1}-4\sqrt{2}\}-3\{-3\sqrt{2}-6\sqrt{2}\}=-4+19\sqrt{2}.$

B1. $\int x\sin x\,dx=x(-\cos x)-\int(-\cos x)\,dx=-x\cos x+\sin x+C.$

WARNING: In the Final Exam, if you forget the "+C" in your answer for an indefinite integral, marks will be deducted !!!

B2. Let $u=2+\ln x$. Then $\int\frac{1}{x(2+\ln x)^3}\,dx=\int\frac{du}{u^3}$

$=\frac{u^{-2}}{-2}+C=\frac{-1}{2(2+\ln x)^2}+C.$

B3. $\int\tan^{78}x\sec^4x\,dx=\int(\tan^{78}x)(1+\tan^2x)\sec^2x\,dx$

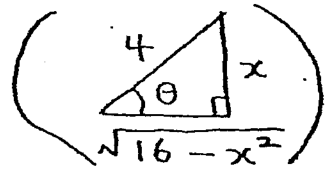
$=\int u^{78}(1+u^2)\,du$ (by letting $u=\tan x$)

$=\int(u^{78}+u^{80})\,du=\frac{1}{79}u^{79}+\frac{1}{81}u^{81}+C=\frac{\tan^{79}x}{79}+\frac{\tan^{81}x}{81}+C$

B4. Let $x = 4\sin\theta$. Then $\displaystyle\int \frac{1}{(16-x^2)^{3/2}}\,dx = \int \frac{4\cos\theta\,d\theta}{(16-16\sin^2\theta)^{3/2}}$

$$= \int \frac{4\cos\theta\,d\theta}{64\cos^3\theta} = \frac{1}{16}\int \sec^2\theta\,d\theta = \frac{1}{16}\tan\theta + C$$

$$= \frac{x}{16\sqrt{16-x^2}} + C$$



B5. $\displaystyle\int \frac{5x^2+4x+2}{x(x^2+1)}\,dx = \int\left(\frac{2}{x} + \frac{3x+4}{x^2+1}\right)dx$ (by Partial Fractions)

$$= 2\ln|x| + \frac{3}{2}\ln|x^2+1| + 4\arctan x + C.$$

B6. Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u\,du = dx$.

Hence $\displaystyle\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u\,du}{1+u} = 2\int \frac{(u+1)-1}{1+u}\,du$

$$= 2\int\left(1 - \frac{1}{1+u}\right)du = 2\left(u - \ln|1+u|\right) + C = 2\left(\sqrt{x} - \ln\left|1+\sqrt{x}\right|\right) + C$$

B7. Let $u = x-2$.

Then $\displaystyle\int_0^4 e^{(x-2)^4}\,dx = \int_{-2}^{2} e^{u^4}\,du = k.$

Hence $\displaystyle\int_0^4 x\,e^{(x-2)^4}\,dx$

$$= \int_{-2}^{2} (u+2)\,e^{u^4}\,du$$

$$= \underbrace{\int_{-2}^{2} u\,e^{u^4}\,du}_{\text{This is 0, since } ue^{u^4} \text{ is an odd function on } [-2,2]} + \underbrace{2\int_{-2}^{2} e^{u^4}\,du}_{\text{This is simply } 2k.}$$

$$= 0 + 2k = \underline{\underline{2k}}$$

NOTE: This question is marked extremely strictly. No credit is awarded unless your solution is completely correct. If you use $u = x-2$ and are able to write $\int e^{u^4}du$ and $\int (u+2)e^{u^4}du$, you will still get zero if you don't change $\int_0^4$ to $\int_{-2}^{2}$. The justification for $\int_{-2}^{2} ue^{u^4}du = 0$ is worth 5 marks. That "odd function on $[-2,2]$" is the crucial step in the solution.