MAT135Y

Answers to Term-test 3 (March 2004)

PART A: 1. $\underline{\underline{-1}}$ 2. $\underline{\underline{6}}$ 3. $\underline{\underline{16}}$ 4. $\underline{\underline{\frac{8}{3}}}$ 5. $\underline{\underline{52}}$ 6. $\underline{\underline{10}}$
7. $\underline{\underline{\frac{2\pi}{15}}}$ 8. $\underline{\underline{\frac{3}{2}}}$ 9. $\underline{\underline{\frac{1}{6}}}$ 10. $\underline{\underline{\frac{1}{2}(e-2)}}$ 11. $\underline{\underline{\frac{5}{6}}}$ 12. $\underline{\underline{\frac{1}{3}\ln 2}}$

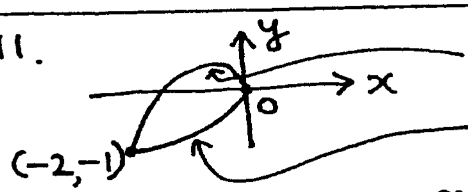
A10. $\displaystyle\int \frac{xe^x}{1+2x+x^2}\,dx = \int \underbrace{xe^x}_{u}\ \underbrace{\frac{1}{(1+x)^2}dx}_{dv} = uv - \int v\,du$

$\displaystyle = (xe^x)\left(\frac{-1}{1+x}\right) - \int\left(\frac{-1}{1+x}\right)(xe^x + e^x)\,dx$

$\displaystyle = \frac{-xe^x}{1+x} + \int\left(\frac{1}{1+x}\right)e^x(x+1)\,dx = \frac{-xe^x}{1+x} + \int e^x\,dx$

$\displaystyle = \frac{-xe^x}{1+x} + e^x + C$. Hence final answer $\displaystyle = \frac{-xe^x}{1+x} + e^x\Big]_0^1 = \underline{\underline{\frac{e-2}{2}}}$.

A11.



$x + x^2 + 2y = 0$ or $y = \dfrac{-x-x^2}{2}$

$x - y + y^2 = 0$ or $y = \dfrac{1-\sqrt{1-4x}}{2}$

$\displaystyle A = \int_{-2}^{0}\left(\frac{-x-x^2}{2} - \frac{1-\sqrt{1-4x}}{2}\right)dx = \frac{5}{6}.$

A12. given limit $\displaystyle = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^2}{i^3+n^3} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i^2 n}{i^3+n^3}$

$\displaystyle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{(i^2n)/n^3}{(i^3+n^3)/n^3}$

$\displaystyle = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\frac{i}{n}\right)^2}{\left(\frac{i}{n}\right)^3+1} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\text{where } f(x) = \frac{x^2}{x^3+1}\right)$

$\displaystyle = \lim_{n\to\infty}\left(\text{Riemann sum of } f(x) = \frac{x^2}{x^3+1} \text{ on } [0,1]\right)$

$\displaystyle = \int_0^1 f(x)\,dx = \int_0^1 \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\ln|x^3+1|\Big]_0^1 = \frac{1}{3}\ln 2.$

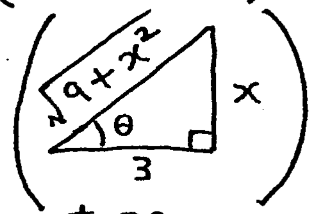
PART B Warning: In the Final Exam, for an indefinite integral, if you forget "+C", marks may be deducted!

B1. $\displaystyle\int xe^{3x}\,dx = x\left(\frac{e^{3x}}{3}\right) - \int\frac{e^{3x}}{3}\,dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C.$

B2. Let $u = \tan(x^2)$. Then $du = 2x\sec^2(x^2)\,dx$.
Hence $\displaystyle\int x\sec^2(x^2)\tan^4(x^2)\,dx = \frac{1}{2}\int u^4\,du$
$\displaystyle = \frac{1}{10}u^5 + C = \frac{1}{10}\tan^5(x^2) + C.$

B3. Let $x = 3\tan\theta$. Then $\displaystyle\int \frac{dx}{(9+x^2)^{3/2}} = \int \frac{3\sec^2\theta\, d\theta}{(9+9\tan^2\theta)^{3/2}}$

$$= \int \frac{3\sec^2\theta\, d\theta}{27\sec^3\theta} = \frac{1}{9}\int \cos\theta\, d\theta$$



$$= \frac{\sin\theta}{9} + C = \frac{x}{9\sqrt{9+x^2}} + C$$

<u>NOTE:</u> If you let $x = 3\sin\theta$, you'll get <u>zero</u>, since that substitution is totally wrong.

B4. $\displaystyle\int \frac{dx}{\sqrt{24-2x-x^2}} = \int \frac{dx}{\sqrt{25-(x+1)^2}} = \arcsin\left(\frac{x+1}{5}\right) + C.$

<u>NOTE:</u> If your "completing the square" is incorrect, you may get very little marks, since you would have changed the question completely.

B5. Integral $\displaystyle = \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1}\right) dx = \int \left(\frac{3}{x} + \frac{-4x-5}{x^2+1}\right) dx$

$= 3\ln|x| - 2\ln|x^2+1| - 5\arctan x + C$. <u>NOTE:</u> You'll get <u>zero</u> if you put down $\frac{A}{x} + \frac{B}{x^2+1}$, since that would be totally wrong.

B6. Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u\,du = dx$.

Hence $\displaystyle\int \frac{dx}{2x^{1/2} + x^{3/2}} = \int \frac{2u\,du}{2u+u^3} = 2\int \frac{du}{2+u^2}$

$$= 2\left\{\frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right)\right\} + C = \sqrt{2}\arctan\left(\sqrt{\frac{x}{2}}\right) + C.$$

B7. $\displaystyle\int \csc^5 x\, dx = \int \underbrace{\csc^3 x}_{u}\, \underbrace{\csc^2 x\, dx}_{dv} = uv - \int v\, du$

$= (\csc^3 x)(-\cot x) - \int (-\cot x)\{3\csc^2 x(-\csc x\cot x)\} dx$

$= -\csc^3 x\cot x - 3\int \csc^3 x\cot^2 x\, dx$

$= -\csc^3 x\cot x - 3\int (\csc^3 x)(\csc^2 x - 1)\, dx$

$= -\csc^3 x\cot x - 3\int \csc^5 x\, dx + 3\int \csc^3 x\, dx.$

Hence $4\int \csc^5 x\, dx = -\csc^3 x\cot x + 3\int \csc^3 x\, dx.$

Hence $\displaystyle\int_{\pi/4}^{\pi/2} \csc^5 x\, dx = \frac{1}{4}\left\{-\csc^3 x\cot x\Big]_{\pi/4}^{\pi/2} + 3\int_{\pi/4}^{\pi/2} \csc^3 x\, dx\right\}$

$$= \frac{1}{4}\left\{-0 + (\sqrt{2})^3\right\} + \frac{3}{4}L = \underline{\frac{1}{4}(2\sqrt{2} + 3L)}$$

<u>NOTE:</u> This question is marked very strictly. You'll get very little marks (probably <u>zero</u>) unless your solution is completely correct.
