MAT 135Y

Answers to Term-test 2 Dec. 2002

1. -2
2. $\frac{1}{2}$
3. $-\frac{2}{3}$
4. $\frac{4}{7}$
5. $-\frac{7}{37}$
6. $2-\frac{2}{\sqrt{3}}$
7. $\frac{25}{4}$
8. -1
9. $-\frac{2}{27}$
10. $(-1, 2)$
11. 8
12. 40 sq cm/sec.
13. $4(1+\ln 2)$
14. $y=2x-1$
15. 5
16. $\frac{2}{e}$
17. $\frac{2}{3}$
18. $2^{74}+2^{46}-2^{30}$
19. $\frac{3}{2}$
20. 2 ft/sec.

Solutions to last 6 questions of Test 2 (Dec. '02)

15. $y = ax^5 \Rightarrow y' = 5ax^4$.

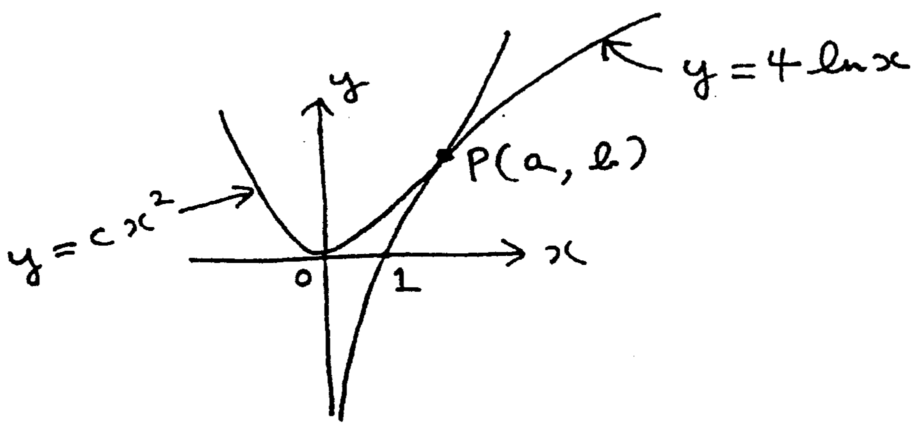
$x^2 + ky^2 = b \Rightarrow 2x + 2kyy' = 0 \Rightarrow y' = \dfrac{-x}{ky}$

$\Rightarrow y' = \dfrac{-x}{kax^5}$ (since $y = ax^5$)

$= \dfrac{-1}{kax^4} = -\dfrac{1}{5ax^4}$ (because of orthogonality)

Hence $k = 5$.

16.



If the two curves have exactly one point in common, say $P(a,b)$, then the two curves must be tangential at P.

$$\left.\begin{array}{l} y = cx^2 \Rightarrow y' = 2cx \Rightarrow y'|_P = 2ca \\ y = 4\ln x \Rightarrow y' = \frac{4}{x} \Rightarrow y'|_P = \frac{4}{a} \end{array}\right\} \Rightarrow 2ca = \frac{4}{a} \Rightarrow c = \frac{2}{a^2}.$$

But y-coordinate of $P = 4\ln a = ca^2$.

Hence $4\ln a = ca^2$

$= \left(\dfrac{2}{a^2}\right) a^2$

$= 2$.

Hence $\ln a = \frac{1}{2}$ and $a = e^{\frac{1}{2}}$.

Hence $c = \dfrac{2}{a^2} = \underline{\underline{\dfrac{2}{e}}}$.

To do this problem, you only need to use L'Hôpital's Rule twice.

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x}\right)$$

$$=\lim_{x\to 0}\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\quad \left(\text{of form } \frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{2\sin x\cos x - 2x\cos^2 x + 2x^2\cos x\sin x}{2x^2\sin x\cos x + 2x\sin^2 x}\quad \left(\text{by L'Hôpital's Rule}\right)$$

$$=\lim_{x\to 0}\frac{\sin x - x\cos x + x^2\sin x}{x^2\left(\frac{1}{2}\sin 2x\right)+x\sin^2 x}\quad \left(\begin{array}{c}\text{after cancelling 2 and}\\ \text{getting rid of } \cos x,\\ \text{since } \lim_{x\to 0}\cos x = 1\end{array}\right)$$

$$=\lim_{x\to 0}\frac{\cos x + (-\cos x + x\sin x) + (2x\sin x + x^2\cos x)}{(x\sin 2x + x^2\cos 2x) + (\sin^2 x + 2x\sin x\cos x)}$$

(by L'Hôpital's Rule)

$$=\lim_{x\to 0}\frac{3x\sin x + x^2\cos x}{2x\sin 2x + x^2\cos 2x + \sin^2 x}$$

$$=\lim_{x\to 0}\frac{\frac{3\sin x}{x}+\cos x}{2\left(\frac{\sin 2x}{x}\right)+\cos 2x + \frac{\sin^2 x}{x^2}}\quad \left(\begin{array}{c}\text{after dividing}\\ \text{top and bottom}\\ \text{by } x^2\end{array}\right)$$

$$=\frac{3+1}{2(2)+1+1}=\frac{4}{6}=\underline{\underline{\frac{2}{3}}}.$$

There are several other ways to do this question!

You can easily show that

$$f(x)=\frac{1}{4}\left(e^{8x}+e^{4x}-1-e^{-4x}\right).$$

Successive differentiations will then give

$$f^{(20)}(x)=2\left(8^{19}e^{8x}\right)+4^{19}e^{4x}-4^{19}e^{-4x}.$$

Hence

$$f^{(20)}(\ln 4)=2^{58}e^{8\ln 4}+2^{38}e^{4\ln 4}-2^{38}e^{-4\ln 4}$$

$$=2^{58}(4^8)+2^{38}(4^4)-2^{38}(4^{-4})$$

$$=2^{74}+2^{46}-2^{30}.$$

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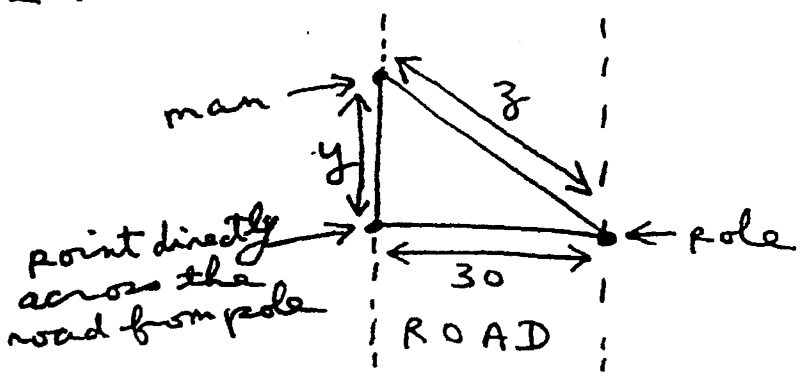
19. It is straight forward to show that the largest cone inscribed inside a sphere of radius R has volume $\frac{32\pi R^3}{81}$.

Since $\frac{32\pi R^3}{81} = \frac{4\pi}{3}$, hence $R = \underline{\underline{\frac{3}{2}}}$.

20.

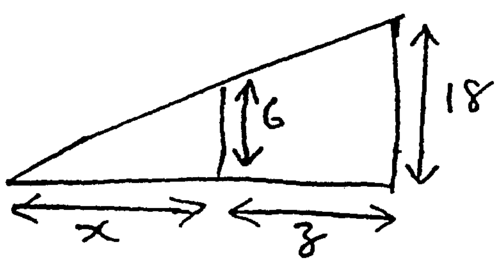


given: $\frac{dy}{dt} = 5$.

$$y^2 + 30^2 = z^2.$$

$$2y\frac{dy}{dt} = 2z\frac{dz}{dt}.$$

So $$\frac{dz}{dt} = \frac{y\frac{dy}{dt}}{z} = \frac{5y}{z}.$$



Length of shadow $= x$.

$$\frac{x+z}{18} = \frac{x}{6}.$$

$$18x = 6x + 6z.$$

$$12x = 6z.$$

$$x = \tfrac{1}{2}z. \text{ So } \frac{dx}{dt} = \frac{1}{2}\frac{dz}{dt} = \frac{1}{2}\left(\frac{5y}{z}\right).$$

When $y = 40$, $z = 50$ and hence $\frac{dx}{dt} = \frac{1}{2}\left(\frac{200}{50}\right) = \underline{\underline{2}}$ ft/sec.
