AN AFFINE BEZOUT TYPE THEOREM AND PROJECTIVE COMPLETIONS OF AFFINE VARIETIES

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Dedicated to my advisor Pierre Milman on the occasion of the first anniversary of our Working Seminar

Abstract. We study projective completions of affine algebraic varieties that are given by filtrations on their rings of regular functions, including a formula for their degrees. For a quasifinite polynomial map P (i.e. with all fibers finite) of affine varieties, we prove that there are completions of the source that do not add points at infinity for P (i.e. in the intersection of completions of the hypersurfaces corresponding to a generic fiber and determined by the component functions of P). Moreover we show that there are 'finite type' completions with the latter property determined by 'degree like functions' that can be expressed as the maximum of a finite number of 'semidegrees', i.e. maps of the ring of regular functions excluding zero, into integers, which send products into sums and sums into maxima (with a possible exception when the summands have the same semidegree). We characterize the latter type filtrations as the ones for which the ideal of the 'hypersurface at infinity' is radical. Moreover, we establish a one-to-one correspondence between the collection of minimal associated primes of the latter ideal and the unique minimal collection of semidegrees needed to define the corresponding degree like function. We also prove an 'affine Bezout type' theorem for quasifinite polynomial maps $P : \mathbb{C}^n \to \mathbb{C}^n$ which admit completions that do not add points at infinity for P and are determined by semidegrees.

Introduction

Throughout this article X will be an irreducible affine algebraic variety of dimension n over $\mathbb{C}$. By a projective completion of X we will mean an open immersion given by an algebraic morphism $\psi : X \hookrightarrow X'$ of X onto a dense open subset of an algebraic subvariety X' of some projective space $\mathbb{P}^N$. Whenever it is clear what the morphism ψ is from the context, we will say that X' is a projective completion of X. It turns out that if the complement of X in X' is a hypersurface, i.e. is the zero set of one homogeneous polynomial, then the map ψ and the space $X^{\mathcal{F}} := X'$ are determined by a *filtration* $\mathcal{F}$ on the ring of regular functions on X. By definition for finitely many closed subvarieties $V_1, \ldots, V_k$ of X, a completion ψ *preserves the intersection of* $V_1, \ldots, V_k$ *at* ∞ if $\bar{V}_1 \cap \cdots \cap \bar{V}_k \cap X_\infty = \emptyset$, where $X_\infty := X' \setminus X$ is the set of 'points at infinity' and $\bar{V}_j$ is the closure of V_j in X' for every j. Given a polynomial map $P = (P_1, \ldots, P_q) : X \to \mathbb{C}^q$, $a = (a_1, \ldots, a_q) \in P(X)$ and a completion

2000 *Mathematics Subject Classification.* 14A10.

Key words and phrases. Completion, Projective completion, Degree like function, semidegree, quasidegree, affine Bezout type, Degree of completion.

The author expresses his gratitude to Professor Pierre Milman for posing the questions and guidance throughout his work. He also thanks the referee for his/her valuable suggestions.

ψ, we say that *ψ preserves a fiber $P^{-1}(a)$ at ∞* if ψ preserves the intersection of the hypersurfaces $H_i(a) := \{x \in X : P_i(x) = a_i\}$, $i = 1, \dots, q$. Finally, when P is quasifinite, i.e. has only finite fibers, we say that completion *ψ preserves P at ∞* provided it preserves generic fibers $P^{-1}(a)$ at ∞, i.e. for all a in a non-empty Zariski open subset of $P(X)$. Completions which preserve quasifinite maps at ∞ are the primary object of study in this article. We are motivated by an objective of an 'affine Bezout type' formula for the size of a generic fiber of any polynomial quasifinite map.

Constructions of a number of beautiful 'affine Bezout type' theorems in [Kus76], [Ber75], [Kho78], [Roj94], [Roj99], [RW96] have one property in common: they all involve completions of affine varieties, which preserve quasifinite polynomial maps at ∞. Here, by making use of [KK08], we provide for $X = \mathbb{C}^n$ and any filtration $\mathcal{F}$ on $\mathbb{C}[x_1, \dots, x_n]$ a description of the degree of an appropriate d-uple embedding of the projective completion $X^{\mathcal{F}}$ in terms of the volume of a convex body determined by $\mathcal{F}$. Moreover, if P is a quasifinite map, $\mathcal{F}$ is determined by a *semidegree* and $X^{\mathcal{F}}$ preserves generic fibers of P at ∞, then we derive (in section 1) an affine Bezout type formula for the size of these generic fibers.

Our article is organized as follows. In the first section we define the notion of a filtration and state a theorem characterizing the completions coming from filtrations. After several basic examples of filtrations and corresponding completions, we give an example of a projective completion of an affine variety $X \neq \mathbb{C}^n$ which does not come from a filtration. On the other hand, we ask the following

Question. Does $\mathbb{C}^n$ admit a projective completion not induced by a filtration?

We continue with a result on the existence of completions coming from filtrations which preserve the intersection at ∞ of a collection of closed subvarieties $V_1, \dots, V_k$ of X such that $\cap_{i=1}^k V_i$ is finite, and those which preserve a given quasifinite map at ∞. Then we present examples of completions determined by semidegrees and *finite type degree like functions*, which for the sake of brevity we call *quasidegrees*. Our examples include classical weighted projective spaces and toric completions of the complex n-torus corresponding to convex integral polytopes. We end this section with statements and examples of the aforementioned result on degrees of completions determined by filtrations and an affine Bezout type theorem. In section 2 we state our main results on projective completions determined by quasidegrees. Our first theorem classifies the filtrations determined by semi- and quasidegrees. As a corollary we deduce that for a completion $\psi : X \hookrightarrow X'$ given by a quasidegree δ, the irreducible components of X_∞ are in a one-to-one correspondence with the unique minimal collection of semidegrees defining δ. Given an arbitrary completion which comes from a filtration and preserves the intersection at ∞ of a collection of subvarieties, we show in our (main) existence theorem that there is a completion determined by a quasidegree which preserves the intersection at ∞ of the subvarieties in the collection. We make use of the latter to conclude the existence of completions determined by quasidegrees which preserve a given quasifinite polynomial map P at ∞. Not all quasifinite maps admit completions of the latter type determined by a quasidegree which is also a semidegree, as our example of section 4 shows. Section 3 we devote to a sketch of a proof of our main existence theorem. Detailed proofs of the results described in this article can be found in [Mon09].

1. Projective completions and an affine Bezout type theorem

From now on A will denote the ring of regular functions on X. A filtration $\mathcal{F}$ on A is a family $\{F_i : i \in \mathbb{Z}\}$ of vector subspaces of A such that

(1) $F_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$,
(2) $1 \in F_0$,
(3) $A = \bigcup_{i\in\mathbb{Z}} F_i$, and
(4) $F_iF_j \subseteq F_{i+j}$ for all i, j.

Filtration $\mathcal{F}$ is called *non-negative* if $F_i = 0$ for all $i < 0$. Associated to each filtration $\mathcal{F}$ there are two graded rings:

$$A^{\mathcal{F}} := \bigoplus_{i\in\mathbb{Z}} F_i \quad \text{and} \quad \mathrm{gr}\, A^{\mathcal{F}} := \bigoplus_{i\in\mathbb{Z}} (F_i/F_{i-1}).$$

We denote a copy of $f \in F_d$ in the d-th graded component of $A^{\mathcal{F}}$ by $(f)_d$. (Multiplication in $A^{\mathcal{F}}$ is given by: $(\sum_d (f_d)_d)(\sum_e (g_e)_e) := \sum_k \sum_{d+e=k} (f_dg_e)_k$.)

We will make use of the following:

(1) associated to a finitely generated $\mathbb{C}$-algebra $S = \bigoplus_{i\geq 0} S_i$ graded over $\mathbb{N}$, there is an algebraic variety Proj S [Har, section II.2];
(2) for an integer $d > 0$, the inclusion of the graded subring $S^{[d]} := \bigoplus_{i\geq 0} S_{id}$ into S induces a map Proj $S \to$ Proj $S^{(d)}$ which is an isomorphism of algebraic varieties. It is called the 'd-uple embedding' of Proj S [Har, exercise II.5.13].

We say that $\mathcal{F}$ is a *finite type* filtration provided $A^{\mathcal{F}}$ is a finitely generated $\mathbb{C}$-algebra. Let $X^{\mathcal{F}} := \mathrm{Proj}\, A^{\mathcal{F}}$. Identifying X with $\mathrm{Spec}\, A$, there is a natural inclusion $\psi_{\mathcal{F}} : X \hookrightarrow X^{\mathcal{F}}$ which takes a prime ideal p of A to the homogeneous prime ideal $\tilde{p} := \bigoplus_{i\in\mathbb{Z}} (p \cap F_d)$ of $A^{\mathcal{F}}$. Our theorem below says that under certain conditions $\psi_{\mathcal{F}}$ is indeed a projective completion of X. Moreover, it gives a characterization of completions corresponding to filtrations up to a d-uple embedding for some $d \geq 1$. Recall that the *homogeneous coordinate ring* of a closed subvariety of an n-dimensional weighted projective space is the factor of the polynomial ring of $n+1$ variables by the ideal of all weighted homogeneous polynomials vanishing on this variety.

Theorem 1.1. *If $\mathcal{F}$ is a non-negative finite type filtration on A such that $F_0 = \mathbb{C}$, then $\psi_{\mathcal{F}}$ is an open immersion of X into $X^{\mathcal{F}}$ and $X^{\mathcal{F}}$ is a closed subvariety of a weighted projective space $\mathbf{W}\mathbb{P}$. $A^{\mathcal{F}}$ is isomorphic as a graded ring to the homogeneous coordinate ring of $X^{\mathcal{F}}$ in $\mathbf{W}\mathbb{P}$. The complement of X in $X^{\mathcal{F}}$ is the hypersurface $V((1)_1)$ and it is isomorphic to Proj (gr $A^{\mathcal{F}}$). Conversely, if $\mathbf{W}\mathbb{P}$ is any weighted projective space and $\psi : X \hookrightarrow X' \subseteq \mathbf{W}\mathbb{P}$ is any projective completion of X such that $X' \setminus X = V(f)$ for some f in the homogeneous coordinate ring of X', then there is a non-negative finite type filtration $\mathcal{F}$ on A and a positive integer d such that $F_0 = \mathbb{C}$, $X' \cong X^{\mathcal{F}}$ and the homogeneous coordinate ring of the d-uple embedding of X is isomorphic as a graded ring to $A^{\mathcal{F}}$.*

In view of Theorem 1.1, we call a filtration $\mathcal{F}$ *complete* if it is non-negative, finite type and $F_0 = \mathbb{C}$. Below are examples of complete filtrations and corresponding completions for $X = \mathbb{C}^n$ and $A = \mathbb{C}[x_1, \ldots, x_n]$.

Examples.

(1) Let $d_1, \dots, d_n$ be any n positive integers and set $d_0 = 1$. Let F_k be the linear span of all monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ such that $\sum_{i=1}^n \alpha_i d_i \leq k$. Then $A^{\mathcal{F}} \cong \mathbb{C}[x_0, \dots, x_n]$ is graded by a weighted degree with weight of x_i being d_i for $i = 0, \dots, n$, and $X^{\mathcal{F}}$ is the classical weighted projective space $\mathbb{P}^n(d_0, d_1, \dots, d_n)$.

(2) Let F_k be the set of polynomials of degree less than or equal to dk, where d is a fixed positive integer. Then $X^{\mathcal{F}}$ is the image of the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^m$, where $m = \binom{n+d-1}{n-1}$. In particular, for $n = 1$, $X^{\mathcal{F}}$ is the so called 'rational canonical curve of degree d'.

(3) Let F_1 be the linear span of all monomials of degree less than or equal to two except for x_n^2. Let $F_k = (F_1)^k$ for $k \geq 1$. Then $X^{\mathcal{F}}$ is the variety resulting from a blow up of $\mathbb{P}^n$ at the point $[0 : \cdots : 0 : 1]$.

Remark 1.

(i) The first part of Theorem 1.1 is well known.

(ii) Not all projective completions of affine varieties are determined by a filtration. Our example below is a variation of an example by Mike Roth and Ravi Vakil considered in [RV04]. Let X' be a nonsingular cubic curve in $\mathbb{P}^2$. Let O be one of its 9 inflection points. Consider the group structure on X' with O as the origin. Pick any point P on X' which is not a torsion point in this group. Then $X := X' \setminus \{P\}$ is an affine variety and the inclusion $i : X \hookrightarrow X'$ is a projective completion. But one can show that there is no homogeneous polynomial f in $\mathbb{C}[x_0, x_1, x_2]$ such that $V(f) \cap X' = \{P\}$. The existence of a polynomial f satisfying the latter property is invariant under all d-uple embeddings ψ_d of X into $\mathbb{P}^{d+1}$. Therefore, the first part of Theorem 1.1 implies that there is no positive integer d such that $\psi_d \circ i$ is the completion corresponding to a filtration.

Our theorem below asserts the existence of completions coming from filtrations which preserve intersections at ∞ of different collections of subvarieties of X:

Theorem 1.2.

(1) *Let $V_1, \dots, V_k$ be closed subvarieties of an affine variety X such that $\bigcap_i V_i$ is a finite set. Then there is a complete filtration $\mathcal{F}$ on A such that $\psi_{\mathcal{F}}$ preserves the intersection of the V_i's at ∞.*

(2) *Let $P = (P_1, \dots, P_q) : X \to Y \subseteq \mathbb{C}^q$ be a quasifinite map of affine varieties. Then there is a complete filtration $\mathcal{F}$ on A such that $\psi_{\mathcal{F}}$ preserves P at ∞.*

Remark 2. Let $X \subseteq \mathbb{C}^p$. Trivial filtrations like the one induced by the projective closure of the graph in $\mathbb{C}^{p+q}$ of map P from (2) of Theorem 1.2 do not necessarily preserve P at ∞, as the following example shows. Let $X = Y = \mathbb{C}^2$ and $P : X \to Y$ be the quasifinite map defined by $P_1 := x_1^3 + x_1^2 x_2 + x_1 x_2^2 - x_2$, and $P_2 := x_1^3 + 2x_1^2 x_2 + x_1 x_2^2 - x_2$. Let Γ be the graph of P in $\mathbb{C}^4$. Choose coordinates (x_1, x_2, y_1, y_2) of $\mathbb{C}^4$ such that $\Gamma = V(y_1 - P_1(x_1, x_2), y_2 - P_2(x_1, x_2))$. Let X' be the closure of Γ in $\mathbb{P}^4$ and let the coordinates of $\mathbb{P}^4$ be $[Z : X_1 : X_2 : Y_1 : Y_2]$ where $x_i = X_i/Z$ and $y_i = Y_i/Z$ for $i = 1, 2$. Identify X with Γ. Then a direct calculation using Puiseux expansions at $[0 : 0 : 1] \in \mathbb{P}^2$ shows that for all $a := (a_1, a_2) \in Y$, point $[0 : 0 : 1 : 0 : 0]$ is in the intersection of the closure in X' of the curves $H_i(a) := \{x \in X : P_i(x) = a_i\}$ for $i = 1, 2$. We thus conclude that completion X' of X does not preserve *any* fiber of P at ∞.

Every filtration $\mathcal{F}$ on A has an associated function $\delta_{\mathcal{F}} : A \to \mathbb{Z} \cup \{-\infty\}$,

$$\delta_{\mathcal{F}}(f) := \min\{d : f \in F_d\}.$$

Filtration $\mathcal{F}$ is uniquely determined by $\delta_{\mathcal{F}}$ via $F_d := \{f : \delta_{\mathcal{F}}(f) \leq d\}$. We will say that $\delta_{\mathcal{F}}$ is non-negative (respectively complete) whenever $\mathcal{F}$ is non-negative (respectively complete). The multiplicative property (i.e. property (4) in the definition) of filtration $\mathcal{F}$ translates into the following in terms of $\delta_{\mathcal{F}}$:

$$(*) \qquad \delta_{\mathcal{F}}(fg) \leq \delta_{\mathcal{F}}(f) + \delta_{\mathcal{F}}(g) \text{ for all } f, g \in A.$$

Definition.

(i) $\delta_{\mathcal{F}}$ is a *semidegree* iff for all $f, g \in A$ we have an equality in $(*)$.

(ii) $\delta_{\mathcal{F}}$ is a *quasidegree* iff there are finitely many semidegrees $\delta_1, \dots, \delta_N$ corresponding respectively to filtrations $\mathcal{F}_1, \dots, \mathcal{F}_N$ such that

$$(**) \qquad \begin{array}{c} \delta_{\mathcal{F}}(f) = \max_{1 \leq i \leq N} \delta_i(f) \text{ for all } f \in A, \\ \text{or, equivalently, } F_j = \bigcap_{i=1}^{N} F_{i,j} \text{ for all } j \in \mathbb{Z}. \end{array}$$

Given a presentation of a quasidegree $\delta_{\mathcal{F}}$ as in $(**)$, by means of getting rid of some δ_i's if necessary, we may assume that each δ_i that appears in $(**)$ is *not redundant*, i.e. for every i, there is an $f \in A$ such that $\delta_i(f) > \delta_j(f)$ for all $j \neq i$. If the latter holds, we say that $(**)$ is a *minimal presentation* of $\delta_{\mathcal{F}}$.

Remark 3. For $f \in A$, by $\langle f \rangle$ we mean the ideal in A generated by f. It is a straightforward consequence of $\delta_{\mathcal{F}}$ being a semidegree that for all $f \in A$, ideals $\bigoplus_{d \in \mathbb{Z}} \langle f \rangle \cap F_d$ and $(f)_{\delta_{\mathcal{F}}(f)} A^{\mathcal{F}}$ of $A^{\mathcal{F}}$ coincide.

Examples.

(4) Weighted degrees on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ are semidegrees. When the weight d_i of x_i is positive for each i, the weighted degree is complete and the associated completion is that of example (1).

(5) Let $\mathcal{F}$ be a filtration on A corresponding to a semidegree $\delta_{\mathcal{F}}$. Pick $h \in A$ and an integer d with $d < \delta_{\mathcal{F}}(h)$. Let $\bar{I}$ be the ideal generated by the class of $(h)_{\delta_{\mathcal{F}}(h)}$ in $\operatorname{gr} A^{\mathcal{F}}$. Consider $R := A[t]$, where t is an indeterminate. Extend $\mathcal{F}$ to a filtration $\mathcal{F}_e$ on R by defining $\delta_{\mathcal{F}_e}(t) = d$. Let $\bar{\mathcal{F}}$ be the filtration on A defined by means of $\delta_{\bar{\mathcal{F}}}(g) := \inf\{\delta_{\mathcal{F}_e}(H) : H \in R, H - g \in \langle t - h \rangle\}$. Then $\delta_{\bar{\mathcal{F}}}$ is a semidegree if and only if $\bar{I}$ is a prime ideal of $\operatorname{gr} A^{\mathcal{F}}$.

(6) Let X be the n-dimensional complex torus $(\mathbb{C}^*)^n$. Then its coordinate ring is $A = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$. Let $\mathcal{P}$ be an integral n dimensional convex polytope (i.e. a polytope in $\mathbb{R}^n$ with vertices in $\mathbb{Z}^n$) such that the origin is in the interior of $\mathcal{P}$. Let

$$\delta(\alpha) := \inf\{1/r | \ r \in \mathbb{Q},\ r > 0,\ r\alpha \in \mathcal{P}\}.$$

There is $k \in \mathbb{N}$ such that $\delta(\alpha) \in \frac{1}{k}\mathbb{N}$ for all $\alpha \in \mathbb{Z}^n$. With this k we define a filtration $\mathcal{F}$ on A by letting $\delta_{\mathcal{F}}(x^\alpha) := k\delta(\alpha)$. Then $\delta_{\mathcal{F}}$ is a complete quasidegree defined by semidegrees which are weighted degrees corresponding to the faces of $\mathcal{P}$. Also, completion $X^{\mathcal{F}}$ of $(\mathbb{C}^*)^n$ is isomorphic to the classical *toric completion* $X_{\mathcal{P}}$ of $(\mathbb{C}^*)^n$ determined by $\mathcal{P}$ [Ful93, secion 3.4]. In fact $X_{\mathcal{P}}$ is the image of the k-uple embedding of $X^{\mathcal{F}}$.

Theorem 1.3 (Affine Bezout Type Theorem)**.** *Let X be an affine variety of dimension n and A be the ring of regular functions on X. Assume $\mathcal{F}$ is a complete* *filtration on A such that $\delta_{\mathcal{F}}$ is a semidegree. Let $(f_1)_{d_1}, \ldots, (f_N)_{d_N}$ be a set of generators of $A^{\mathcal{F}}$ and d be N times a common multiple of the d_j's. Denote by D the degree of the d-uple embedding of $X^{\mathcal{F}}$ in the projectivization of the d-th graded component F_d of $A^{\mathcal{F}}$. Let $P = (P_1, \ldots, P_n) : X \to \mathbb{C}^n$ be any quasifinite map. Then for all $a \in \mathbb{C}^n$,*

$$|P^{-1}(a)| \leq \frac{D}{d^n} \prod_{i=1}^{n} \delta_{\mathcal{F}}(P_i), \tag{A}$$

where $|P^{-1}(a)|$ is the size of the fiber $P^{-1}(a)$ counted with multiplicity. If in addition $\psi_{\mathcal{F}}$ preserves fiber $P^{-1}(a)$ at ∞, then (A) holds with an equality.

Let $X = \mathbb{C}^n$ and ν be a 'monomial valuation' on $A = \mathbb{C}[x_1, \ldots, x_n]$, which assigns to $\sum a_\alpha x^\alpha \in A \setminus \{0\}$ the minimal (lexicographically) exponent $\alpha = (\alpha_1, \ldots, \alpha_n)$ among α with $a_\alpha \neq 0$. Let $\mathcal{F}$ be a complete filtration on A, and let d and D be as in Theorem 1.3. Following [KK08], we can associate a convex body Δ to $\mathcal{F}$ such that D is $n!$ times the n-dimensional volume $V_n(\Delta)$ of Δ, namely:

Proposition 1.4. *Let C be the smallest closed cone in $\mathbb{R}^{n+1}$ containing*

$$G := \{(\frac{1}{d}\delta_{\mathcal{F}}(f), \nu(f)) \in \mathbb{Z}_+^{n+1} : f \in A\}.$$

Let Δ be the convex hull of the cross-section of C at the first coordinate value 1. Then $D = n! V_n(\Delta)$.

Proof. Let $L := F_d = \{f \in A : \delta_{\mathcal{F}}(f) \leq d\}$. Let $\mathbb{P}_L$ be the projectivization of L, and $\phi_L : X^{\mathcal{F}} \hookrightarrow \mathbb{P}_L$ be the d-uple embedding. Degree D of $\phi_L(X^{\mathcal{F}})$ in $\mathbb{P}_L$ is the number of common zeros of n generic elements of L, and it is precisely the intersection index $[L, \ldots, L]$ of n copies of L as defined in [KK08]. Since the mapping degree of ϕ_L is 1, the proposition follows from the main theorem of [KK08]. □

Examples.

(7) Let $\delta_{\mathcal{F}}$ be a weighted degree on A corresponding to positive integers $d_1, \ldots, d_n$ as in example (4). Then $(1)_1, (x_1)_{d_1}, \ldots, (x_n)_{d_n}$ is a set of generators of $A^{\mathcal{F}}$ and, with $d = nd_1 \cdots d_n$, convex set $\Delta = \{(1, x) \in \mathbb{R}_+^{n+1} : \sum_{j=1}^n x_j d_j \leq d\}$ and $V_n(\Delta) = \frac{1}{n!} \prod_{j=1}^n \frac{d}{d_j}$. (Since $\psi_{\mathcal{F}}$ preserves all fibers of the identity map $\mathbb{I}$ of $\mathbb{C}^n$ at ∞, formula (A) for $\mathbb{I}$ implies $D = \frac{d^n}{d_1 \cdots d_n}$ directly.) Hence, the right hand side of (A) is $\frac{\prod_j \delta_{\mathcal{F}}(P_j)}{\prod_j d_j}$ and the result is the well known weighted version of Bezout's theorem (see e. g. [Dam99]).

(8) Let $n = 2$ and $\delta_{\mathcal{F}}$ be the weighted degree on $A = \mathbb{C}[x_1, x_2]$ that assigns weight 3 to x_1 and 2 to x_2. Then $\operatorname{gr} A^{\mathcal{F}} \cong \mathbb{C}[x_1, x_2]$ via the map that takes the class of $(f)_{\delta_{\mathcal{F}}(f)}$ to the leading weighted homogeneous component of f. Consider $h := x_1^2 - x_2^3 \in A$. It is easy to see that the ideal generated by the class of $(h)_{\delta_{\mathcal{F}}(h)}$ is prime in $\operatorname{gr} A^{\mathcal{F}}$. Hence, applying the construction of example (5) for this h, we obtain a filtration $\bar{\mathcal{F}}$ on A such that $\delta_{\bar{\mathcal{F}}}$ is a semidegree which takes value 3 on x_1, 2 on x_2 and 1 on $x_1^2 - x_2^3$. It is easy to show that $\bar{\mathcal{F}}$ preserves all fibers of the map $P := (x_2, x_1^2 - x_2^3) : \mathbb{C}^2 \to \mathbb{C}^2$. Since P has only 2 solutions in $\mathbb{C}^2$, applying equation (A) to P it follows that $\frac{D}{d^2} = 1$. Finally, for any $k > 0$, $\bar{\mathcal{F}}$ preserves all fibers of the map $Q_k := (x_1 + (x_1^2 - x_2^3)^2, (x_1^2 - x_2^3)^k) : \mathbb{C}^2 \to \mathbb{C}^2$. Hence, it follows from (A) that for any $k > 0$, the size of every fiber of Q_k is $3k$, which is much smaller

than the estimate $12k$ predicted by the weighted homogeneous version of Bezout's theorem.

(9) In general, assume δ is a semidegree on $\mathbb{C}[x_1, \ldots, x_n]$ that is constructed by repeating the process of example (5) k times starting with a weighted degree δ_0 and for each $i = 1, \ldots, k$, constructing a semidegree δ_i by finding a polynomial h_i whose 'leading term' generates a prime ideal and associating to it weight $d_i < \delta_{i-1}(h_i)$. Then $\frac{D}{d^n} = \frac{\delta_0(h_1)\cdots\delta_{k-1}(h_k)}{\delta_0(x_1)\cdots\delta_0(x_n)d_1\cdots d_k}$ (see [Mon09]).

2. Main Results on Projective Completions

Our theorem below provides a characterization of semi- and quasidegrees in terms of the ideal of $A^{\mathcal{F}}$ corresponding to the 'hypersurface at infinity' of $X^{\mathcal{F}}$.

Theorem 2.1. *Let I be the ideal of $A^{\mathcal{F}}$ generated by $(1)_1$. Then*

(1) *$\delta_{\mathcal{F}}$ is a semidegree if and only if I is a prime ideal, and*

(2) *$\delta_{\mathcal{F}}$ is a quasidegree if and only if ideal I admits a prime decomposition. In particular, if $A^{\mathcal{F}}$ is noetherian, then $\delta_{\mathcal{F}}$ is a quasidegree if and only if I is a radical ideal.*

Using the uniqueness of a minimal primary decomposition of a decomposable radical ideal [AM69, chapter 4], we deduce as a consequence the uniqueness of a minimal presentation of a quasidegree. Moreover, when $\mathcal{F}$ is also non-negative, we derive a one-to-one correspondence between the semidegrees in the minimal presentation of $\delta_{\mathcal{F}}$ and the irreducible components of the 'infinite part' X_∞ of $X^{\mathcal{F}}$.

Corollary 2.1.1. *Let $\delta_{\mathcal{F}}$ be a quasidegree. Then*

(1) *there exist unique semidegrees $\delta_1, \ldots, \delta_N$ corresponding to filtrations $\mathcal{F}_1, \ldots, \mathcal{F}_N$ respectively such that $\delta_{\mathcal{F}} = \max_{i=1}^N \delta_i$ is a minimal presentation of $\delta_{\mathcal{F}}$.*

(2) *If $\mathcal{F}$ is also non-negative, then there is a one-to-one correspondence between the irreducible components of $X^{\mathcal{F}} \setminus X$ and the semidegrees in the minimal presentation of $\delta_{\mathcal{F}}$.*

For normal varieties, we additionally prove the following.

Proposition 2.2. *If X is a normal affine variety, and $\delta_{\mathcal{F}}$ is a quasidegree on A, then $A^{\mathcal{F}}$ is integrally closed. In particular, if $\mathcal{F}$ is complete and $\delta_{\mathcal{F}}$ is a quasidegree, then $X^{\mathcal{F}}$ is* projectively normal [Har, exercise II.5.14].

Given a filtration $\mathcal{F}$ on A and an integer $k \geq 0$, let $k\mathcal{F}$ be the filtration such that $\delta_{k\mathcal{F}} := k\delta_{\mathcal{F}}$. We say that another filtration $\mathcal{G}$ on A is *integral over* $\mathcal{F}$ if

(i) $\delta_{\mathcal{G}} \leq \delta_{\mathcal{F}}$, so that there is a natural inclusion $A^{\mathcal{F}} \hookrightarrow A^{\mathcal{G}}$, and

(ii) $A^{\mathcal{G}}$ is integral over $A^{\mathcal{F}}$ under the above inclusion.

Finally, if both $\mathcal{F}$ and $\mathcal{G}$ are non-negative, we say that $\mathcal{G}$ *preserves the intersections at ∞ for the completion determined by $\mathcal{F}$* (in short, *for $\mathcal{F}$*) if and only if for any finite number of closed subsets $X_1, \ldots, X_k$ of X such that the closures of X_i in $X^{\mathcal{F}}$ do not intersect at any point at infinity, the closures of X_i in $X^{\mathcal{G}}$ also do not intersect at any point at infinity. For non-negative filtrations $\mathcal{F}$ and $\mathcal{G}$ on A, it follows that if $\mathcal{G}$ is integral over $\mathcal{F}$ or if $\mathcal{G} = k\mathcal{F}$ for some integer $k > 0$, then $\mathcal{G}$ preserves the intersections at ∞ for $\mathcal{F}$.

Theorem 2.3 (Main Existence Theorem)**.** *Let $\mathcal{F} = \{F_j : j \in \mathbb{Z}\}$ be a finite type filtration on A. Then there is a finite type filtration $\tilde{\mathcal{F}}$ on A and a positive integer* *e such that $\delta_{\tilde{\mathcal{F}}}$ is a quasidegree and $\tilde{\mathcal{F}}$ is integral over $e\mathcal{F}$. If $\mathcal{F}$ is non-negative, then $\tilde{\mathcal{F}}$ is also non-negative, and $\tilde{\mathcal{F}}$ preserves the intersections at ∞ for $\mathcal{F}$.*

As a corollary of theorems (2.3) and (1.2) we deduce the existence of completions determined by quasidegrees which preserve the intersections at ∞ of different collections of subvarieties of X:

Corollary 2.3.1. *Let X be an affine variety of dimension n and A be the ring of regular functions on X.*

(1) *Let $V_1, \ldots, V_k$ be closed subvarieties of X such that $\bigcap_i V_i$ is a finite set. Then there is a complete filtration $\mathcal{F}$ on A such that $\delta_{\mathcal{F}}$ is a quasidegree and $\psi_{\mathcal{F}}$ preserves the intersection of the V_i's at ∞.*
(2) *Let $P = (P_1, \ldots, P_q) : X \to Y \subseteq \mathbb{C}^q$ be a quasifinite map of affine varieties. Then there is a complete filtration $\mathcal{F}$ on A such that $\delta_{\mathcal{F}}$ is a quasidegree and $\psi_{\mathcal{F}}$ preserves P at ∞.*

3. Proof of the Main Existence Theorem

Below we make use of the theory of *Rees' valuations*. We start with the definitions and results on Rees' valuations that we need (see [McA83, chapter XI]). For an ideal I of a ring R let $\nu_I : R \to \mathbb{N} \cup \{\infty\}$ and $\bar{\nu}_I : R \to \mathbb{Q}_+ \cup \{\infty\}$ be defined by: $\nu_I(x) := \sup\{m : x \in I^m\}$ and $\bar{\nu}_I(x) := \lim_{m\to\infty} \frac{\nu_I(x^m)}{m}$. The following is due to Rees [McA83, propositions 11.1, 11.5, corollary 11.6]:

Theorem (Rees)**.** *For any ring R and any ideal I of R, $\bar{\nu}_I$ is well defined. If R is a noetherian domain then*

(1) *there is a positive integer e such that for all $x \in R$, $\bar{\nu}_I(x) \in \frac{1}{e}\mathbb{N}$, and*
(2) *if $k \geq 0$ is an integer then $\bar{\nu}_I(x) \geq k$ if and only if $x \in \bar{I^k}$, where $\bar{I^k}$ is the integral closure of I^k in R.*

Note, that $A^{\mathcal{F}} \cong \bigoplus_{i\in\mathbb{Z}} F_i t^i$, where t is an indeterminate. Under this natural isomorphism, $(f)_i$ is mapped to ft^i and, in particular, $(1)_1$ is mapped to t. Let I be the ideal in $A^{\mathcal{F}}$ generated by $(1)_1$. With ν_I and $\bar{\nu}_I$ from above, let e be a positive integer provided by Rees' theorem such that for all $F \in A^{\mathcal{F}}$, $\bar{\nu}_I(F) \in \frac{1}{e}\mathbb{N}$.

Below, in abuse of notation, we write δ for $\delta_{\mathcal{F}}$. Fix $f \in A$ and $m \in \mathbb{N}$. Then $\delta(f^m) \leq m\delta(f)$. Moreover, since I is generated by $(1)_1$, it follows that $k := m\delta(f) - \delta(f^m)$ is the largest integer such that $(f^m)_{m\delta(f)} \in I^k$. Thus, by definition of ν_I, $\nu_I(((f)_{\delta(f)})^m) = k = m\delta(f) - \delta(f^m)$. Therefore $\delta(f^m)/m = \delta(f) - \nu_I(((f)_{\delta(f)})^m)/m$. It follows that $\bar{\delta}(f) := \lim_{m\to\infty} \delta(f^m)/m$ is well defined and equals $\delta(f) - \bar{\nu}_I((f)_{\delta(f)})$.

For $m \in \mathbb{Z}$, let $\bar{F}_{\frac{m}{e}} := \{f \in A : \bar{\delta}(f) \leq \frac{m}{e}\}$, and consider ring $A^{\bar{\mathcal{F}}} := \bigoplus_{m\in\mathbb{Z}} \bar{F}_{\frac{m}{e}} t^{\frac{m}{e}}$. Since $\bar{\delta} \leq \delta$, it follows that $F_k \subseteq \bar{F}_k$ for each $k \in \mathbb{Z}$. Therefore $A^{\mathcal{F}} \subseteq A^{\bar{\mathcal{F}}}$.

Claim. *$A^{\bar{\mathcal{F}}}$ is integral over $A^{\mathcal{F}}$.*

Proof. It suffices to show that for each $f \in A$, $(f)_{\bar{\delta}(f)}$ is integral over $A^{\mathcal{F}}$. Pick $f \in A$. Then $(f)_{\bar{\delta}(f)}$ is integral over $A^{\mathcal{F}}$ if and only if $\bar{F} := ((f)_{\bar{\delta}(f)})^e$ is integral over $A^{\mathcal{F}}$. From construction of $\bar{\delta}$ it follows that $\bar{F} = (f^e)_{e\bar{\delta}(f)} = (f^e)_{\bar{\delta}(f^e)}$. Let

$F := (f^e)_{\delta(f^e)} \in A^{\mathcal{F}}$ and $\bar{k} := \bar{\nu}_I(F)$. Then $\bar{k} = \delta(f^e) - \bar{\delta}(f^e) = \delta(f^e) - e\bar{\delta}(f)$. It follows that $\bar{k}$ is an integer. Hence by Rees' theorem, F is in the integral closure of $I^{\bar{k}}$ in $A^{\mathcal{F}}$, i.e. F satisfies an equation of the form $F^s + G_1F^{s-1} + \cdots + G_s = 0$, where $G_i \in I^{i\bar{k}}$ for each i. Since $A^{\bar{\mathcal{F}}}$ is a graded ring, we may assume without loss of generality that the degrees of G_i are $i\delta(f^e)$. Then $G_i = (g_i)_{i(\delta(f^e)-\bar{k})}(1)_{i\bar{k}}$ for some $g_i \in A$ with $\delta(g_i) \leq i(\delta(f^e) - \bar{k}) = i\bar{\delta}(f^e)$, $1 \leq i \leq s$. Moreover, in the ring $A^{\bar{\mathcal{F}}}$, $F = (f^e)_{\delta(f^e)} = (f^e)_{\bar{\delta}(f^e)+\bar{k}} = (f^e)_{\bar{\delta}(f^e)}(1)_{\bar{k}} = (1)_{\bar{k}}\bar{F}$. Substituting these values of F and G_i into the equation of integral dependence for F and then cancelling a factor of $(1)_{s\bar{k}}$ we conclude that $(\bar{F})^s + \sum_{i=1}^{s}(g_i)_{i\bar{\delta}(f^e)}(\bar{F})^{s-i} = 0$. Thus $\bar{F}$ is integral over $A^{\mathcal{F}}$, which completes the proof of the claim. □

It follows from the preceeding claim that $A^{\bar{\mathcal{F}}}$ is an $A^{\mathcal{F}}$-submodule of the integral closure B of $A^{\mathcal{F}}$ in $A^{\mathcal{F}}[t^{\frac{1}{e}}, t^{-\frac{1}{e}}]$. Since B is a finite $A^{\mathcal{F}}$ module, and $A^{\mathcal{F}}$ is a finitely generated $\mathbb{C}$-algebra, it follows that B is a noetherian $A^{\mathcal{F}}$-module. Therefore $A^{\bar{\mathcal{F}}}$ is a finitely generated $A^{\mathcal{F}}$-module, and hence is a finitely generated $\mathbb{C}$-algebra. Consider the filtration $\tilde{\mathcal{F}}$ on A such that $\delta_{\tilde{\mathcal{F}}} := e\bar{\delta}$. It is not difficult to show (see [Mon09]) that $\tilde{\mathcal{F}}$ is integral over $e\mathcal{F}$ and $A^{\tilde{\mathcal{F}}}$ is a finitely generated $\mathbb{C}$-algebra. By construction $\delta_{\tilde{\mathcal{F}}}(f^m) = m\delta_{\tilde{\mathcal{F}}}(f)$ for all f and m and therefore the ideal generated by $(1)_1$ in $A^{\tilde{\mathcal{F}}}$ is radical. Hence Theorem 2.1 implies that $\delta_{\tilde{\mathcal{F}}}$ is a quasidegree. This completes the proof of the first assertion of Theorem 2.3. If $\mathcal{F}$ is non-negative, then by construction $\tilde{\mathcal{F}}$ is also non-negative, and applying the remark preceeding Theorem 2.3, we deduce the second assertion.

4. A Quasifinite Map with Points at Infinity for any Semidegree

Let $X = Y = \mathbb{C}^2$ and $P := ((x_1^2 - x_2^4)^2 + x_1x_2, (x_1^2 - x_2^4)^3 + x_1x_2) : X \to Y$. Then P is a quasifinite map and one can show that there is *no* filtration $\mathcal{F}$ determined by a semidegree on $A = \mathbb{C}[x_1, x_2]$ such that $\psi_{\mathcal{F}}$ preserves P at ∞. In fact for each semidegree $\delta_{\mathcal{F}}$ on A, $\delta_{\mathcal{F}}$ does *not* preserve the intersection at ∞ of the pair of curves $V(P_1 - y_1)$ and $V(P_2 - y_2)$, for all $(y_1, y_2) \in Y$. This shows that none of the assertions of corollary (2.3.1) would remain valid if we replace in its conclusion the 'quasidegree' by a 'semidegree'. Note that P of this example is also degenerate in the sense of Bernstein-Kushnirenko-Khovanskii.

Finally, we define a filtration $\mathcal{F}$ corresponding to a quasidegree such that for each $(y_1, y_2) \in Y$, $\psi_{\mathcal{F}}$ preserves the intersection of $V(P_1 - y_1)$ and $V(P_2 - y_2)$ at ∞, and hence preserves *every* fiber of P at ∞. Namely, let $\mathcal{F} := \{F_d : d \geq 0\}$ where $F_0 := \mathbb{C}$, $F_1 := \mathbb{C}\langle 1, x_2, x_1^2 - x_2^4\rangle$, $F_2 := (F_1)^2 + \mathbb{C}\langle x_1\rangle$, and $F_d := \sum_{j=1}^{d-1} F_jF_{d-j}$ for $d \geq 2$. Then $\delta_{\mathcal{F}}$ is a quasidegree with the minimal presentation $\delta_{\mathcal{F}} = \max\{\delta_1, \delta_2\}$, where δ_1 is the weighted degree on A that assigns weight 1 to x_2 and -1 to $x_1 - x_2^2$, and δ_2 is the weighted degree on A that assigns weight 1 to x_2 and -1 to $x_1 + x_2^2$. A direct calculation shows that $\mathcal{F}$ is as promised (see details in [Mon09]).

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