Projective completions of affine varieties via degree-like functions

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Abstract

We introduce and study a class of projective completions of affine algebraic varieties which generalize the construction of toric varieties from convex rational polytopes.

Contents

1 Introduction

Our motivation to study completions via degree like functions is rooted in our project to understand 'affine Bezout-type' theorems. The usual version of Bezout theorem gives a formula for the number of solutions (counted with multiplicities) of n polynomials on the *projective* space $\mathbb{P}^n(\mathbb{K})$ as the product of the degrees of these polynomials. On the contrary, affine Bezout-type theorems estimate the number of solutions of a system of equations on an *affine* variety. In a forthcoming continuation of this article (announced in [15], see also [16, Chapter 3]), we present a general framework for affine Bezout-type theorems based on the theory developed here. In particular, we introduce in the forthcoming work a special class of projective completions that do not 'add solutions at infinity' and which are determined by 'degree-like functions'. The 'normalization' of degree-like functions (Theorem 5.12) implies that for the purpose of adding no solutions at infinity, it suffices to consider only 'subdegrees'. But we must restrict the degree-like functions to a smaller subclass in order to achieve the desired formulae. (Our ultimate dream is a constructive 'affine Bezout-type' theorem estimating the size of a generic fiber of a polynomial mapping of $\mathbb{C}^n$ into itself with a constant nonzero jacobian.) We introduce in the forthcoming article a special class of semidegrees called *iterated semidegrees* - an example of iterated semidgrees on $\mathbb{K}^2$ is given here (Example 3.3). The degree of a completion (of an affine variety) corresponding to an iterated semidegree can be explicitly calculated and this in turn implies that the number of solutions of a system of equations *preserved* by an iterated semidegree can also be explicitly computed. Our hope is that subdegrees corresponding only to iterated semidegrees are sufficient for affine Bezout-type theorems, which would make our theory constructive and bring it closer to our ultimate dream.

1.1 Main results and hopes - briefly

Main Tool: A well known way of constructing projective completions of affine varieties is via filtrations, or equivalently, 'degree-like functions' on their coordinate rings. We study a special class of degree-like functions called 'subdegrees'. A subdegree is by definition the maximum of finitely many 'semidegrees', the latter being degree-like functions which send products into sums.

Main Results: 1. We characterize the completions determined by subdegrees as the ones for which ideal I of the 'hypersurface at infinity' is radical and establish a one-to-one correspondence between the collection of minimal associated primes of I and the unique minimal collection of non-trivial semidegrees needed to define the corresponding subdegree.

2. We show that the completions determined by subdegrees are *non-singular in codimension one at infinity* and establish a relation between the associated semidegrees with the orders of vanishing along the components at the hypersurface at infinity.

3. We introduce the notion of the *normalization at infinity* of completions of affine varieties. Given an arbitrary completion of an affine variety via a degree-like function, we construct a subdegree which normalizes the completion at infinity.

4. In dimension 2, we show (analogous to Gordan's lemma in convex geometry) that a subdegree corresponding to 'finitely generated' and 'complete' semidegrees is also 'finitely generated'.

5. We prove a formula for the pull-back (under a dominating morphism) of the 'divisor at infinity' on completions determined by subdegrees and define the *linking number at infinity* of two semidegrees (which is the inverse of the 'usual' linking numbers of corresponding valuations). We show that in dimension 2, the matrix of linking numbers at infinity of a finite collection of complete semidegrees is invertible.

To advance our investigation towards its 'ultimate dream' it is essential to clarify the validity of the following

Conjecture. *1. Subdegrees determined by 'finitely generated' and 'complete' semidegrees (or something like 'iterated semidegrees') are also 'finitely generated' (in all dimensions).*

2. For any polynomial map of $\mathbb{C}^n$ into itself with generically finite fibers there is a projective completion that does not 'add points at infinity' for generic fibers of this map and corresponds to a finitely generated subdegree determined by (constructively identified) iterated (or something like iterated) semidegrees.

1.2 Main results - an informal discussion

Construction: A familiar construction of completions of affine algebraic varieties is via *filtrations* on their coordinate rings: let X be an arbitrary affine variety over an algebraically closed field $\mathbb{K}$ and $\mathcal{F} = \{F_d : d \geq 0\}$ be a filtration on the coordinate ring $\mathbb{K}[X]$ of X (which in our case means that $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$ is a sequence of vector subspaces of $\mathbb{K}[X]$ such that $\mathbb{K}[X] = \bigcup_{d \geq 0} F_d$, and $F_d F_e \subseteq F_{d+e}$). Then (under certain natural assumptions) $\operatorname{Proj} \bigoplus_{d \geq 0} F_d$ is a *completion* of X, i.e. a complete variety which contains X as a dense open subset (cf. Proposition 2.2).

Giving a filtration on $\mathbb{K}[X]$, on the other hand, is equivalent to defining a *degree-like function* $\delta : \mathbb{K}[X] \to \mathbb{Z}$ which satisfies the following properties:

1. $\delta(f+g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in \mathbb{K}[X]$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$.
2. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in \mathbb{K}[X]$.

The vector spaces $F_d := \{f \in \mathbb{K}[X] : \delta(f) \leq d\}$ define a filtration on $\mathbb{K}[X]$ associated with δ and from that filtration, one constructs a completion X^δ of X as in the first paragraph. For $X = \mathbb{K}^n$ and δ equal to the usual degree of polynomials, the completion of $\mathbb{K}^n$ we get via this construction is the *standard projective space* $\mathbb{P}^n(\mathbb{K})$. If we take δ to be a more general *weighted degree*, we end up with the corresponding *weighted projective space.*

Both the usual and weighted degrees satisfy property 2 with exact *equality* instead of the inequality. We call the degree-like functions which have this property *semidegrees.* A classical example of a class of degree-like functions which are not semidegrees comes from *toric geometry* - where one associates a normal n-dimensional projective variety to a convex integral polytope of dimension n. Each facet (i.e. codimension one face) of such a polytope $\mathcal{P}$ defines a weighted degree on $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. It turns out (see Example 3.4) that the toric variety $X_{\mathcal{P}}$ associated to $\mathcal{P}$ is precisely the completion of the n-torus $(\mathbb{K}^*)^n$ corresponding to the degree-like function $\delta_{\mathcal{P}}$ which is the maximum of the weighted degrees corresponding to the facets of $\mathcal{P}$. In our terminology, $\delta_{\mathcal{P}}$ is an example of a *subdegree* - a degree-like function which is the maximum of finitely many semidegrees.

Guiding principle: generalization of toric completions by means of subdegrees determined by semidegrees. One of the guiding principles of this work is the conviction that the completion of $\mathbb{K}^n$ coming from a semidegree should 'behave similarly' to the weighted projective spaces, which are completions of $\mathbb{K}^n$ corresponding to weighted degrees. As the analogue of the $\delta_{\mathcal{P}}$ corresponding to a polytope $\mathcal{P}$, the same principle leads us to consider subdegrees. The purpose of this article is essentially to put forth some evidence in favour of this principle.

Our first main result is Theorem 4.1, where we show that the *minimal* presentation of a subdegree δ (on the coordinate ring $\mathbb{K}[X]$ of an arbitrary affine variety X) as the maximum of finitely many semidegrees is unique. Moreover, in the corresponding completion X^δ of X, the irreducible components of the *hypersurface at infinity* $X_\infty := X^\delta \setminus X$ have a canonical one-to-one correspondence with the non-trivial semidegrees associated to δ, in the same way that the irreducible components of $X_{\mathcal{P}} \setminus (\mathbb{K}^*)^n$ correspond to the facets of $\mathcal{P}$.

It turns out that X^δ is *relatively normal at infinity with respect to* X (Proposition 5.7), which means that for all open subset U of X^δ, the ring of regular functions on U is integrally closed in the ring of functions which are regular on $X \cap U$. A consequence (which we in fact prove earlier!) of it is that X^δ is *non-singular in codimension one at infinity* (Proposition 5.1), i.e. the codimension in X^δ of $X_\infty \cap \operatorname{Sing} X^\delta$ is at least two (where $\operatorname{Sing} X^\delta$ is the set of singular points of X^δ). Given a completion $X \hookrightarrow X^\delta$ determined by an arbitrary degree-like function δ, we introduce a 'normalization' procedure to produce a subdegree $\tilde{\delta}$ and a finite morphism $\phi : X^{\tilde{\delta}} \to X^\delta$ such that ϕ is identity on X (Theorem 5.12). Moreover, $X^{\tilde{\delta}}$ has the following universal property: if $\psi : Y \to X^\delta$ is a dominant morphism of algebraic varieties such that Y is *relatively normal at infinity with respect to* X (via ψ), then there is a unique lift of ψ to $X^{\tilde{\delta}}$, i.e. there is a commuting diagram as follows:

$$\begin{tikzcd}
 & X^{\tilde{\delta}} \arrow[d, "\phi"] \\
Y \arrow[ru, "\exists!"] \arrow[r, "\psi"] & X^\delta
\end{tikzcd}$$

We say that $X^{\tilde{\delta}}$ is the *normalization of* X^δ *at infinity (with respect to* X*).* In particular, when X is normal, then $X^{\tilde{\delta}}$ is the normalization of X^δ. The construction of $\tilde{\delta}$ from δ generalizes the well known procedure of constructing the normalization of a non-normal toric variety (determined by a finite subset of a lattice) by 'filling the holes' [4, Theorem 3.A.5].

Semidegrees contain more information than orders of vanishing along components of the hypersurface at infinity: The negative of semidegrees satisfy the multiplicative and additive properties of *valuations.* Indeed, if η is a semidegree and d_η is the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\eta(f) : f \in \mathbb{K}[X]\}$, then $-\frac{\eta}{d_\eta}$ is a discrete valuation on $\mathbb{K}[X]$. In particular, we show that if δ is a subdegree with associated semidegrees $\delta_1, \ldots, \delta_N$, then for each i, $1 \leq i \leq N$, $\eta_i := -\frac{\delta_i}{d_{\delta_i}}$ is the order of vanishing along the component of the hypersurface at infinity of X^δ associated to δ_i (Proposition 5.1). Curiously, δ_i's in general contain more information about the completion X^δ than η_i's, i.e. the orders of vanishing along the components of the hypersurface at infinity of X^δ do *not* in general determine

X^δ. In particular, for $n \geq 3$, there are *non-isomorphic* toric completions of $\mathbb{K}^n$ with identical sets of discrete valuations along the components of the hypersurfaces at infinity (Example 3.5).

The data contained in the associated semidegrees $\delta_1, \ldots, \delta_N$ of a subdegree δ that get lost when considering only the orders of vanishing along the components of the hypersurface at infinity are precisely the integers d_{δ_j}, $1 \leq j \leq N$ (in the notation of the previous paragraph). These numbers also appear in a more 'natural' setting - they are inversely proportional to the coefficients of irreducible Weil divisors in the expansion for the *divisor at infinity.* The divisor at infinity is an analogue of the *divisor of a polyhedron* in toric geometry (see, e.g. [4, Section 7.1]). We prove a formula (in Proposition 5.18) for the 'infinite part' of the pull-back of the divisor at infinity under a dominant morphism between completions of affine varieties corresponding to subdegrees. In connection with defining the pull-back of Weil divisors under birational regular mappings, P. Samuel introduced in [19] the notion of *linking numbers* of two discrete valuations. In [11] the definition of linking numbers was generalized to the case of *pseudo-valuations.* The coefficients of the irreducible Weil divisors in our pull-back formula turn out to be the inverses of the linking numbers (in the sense of [11]) of the (negative of the) corresponding degree-like functions. We refer to these coefficients as the *linking numbers at infinity* of corresponding degree-like functions.

Finite generation of degree-like functions: Let A be a finitely generated algebra over $\mathbb{K}$. A degree-like function on A is called *finitely generated* if the corresponding graded ring is also a finitely generated algebra over $\mathbb{K}$. A basic building block of the theory of toric varieties is *Gordan's lemma* [6, Proposition 1, Section 1.2], which says that the semigroup of integral points in a convex rational cone in $\mathbb{R}^n$ is finitely generated. Another equivalent formulation of Gordan's lemma is that the maximum of finitely many weighted degrees (in $(x_1, \ldots, x_n)$) on $\mathbb{K}[x_1, \ldots, x_n]$ is finitely generated. The analogous question, which arises naturally in the context of the structure theorem of subdegrees (theorem 4.1) is the following:

Question 1. Is the maximum of finitely many finitely generated semidegrees also finitely generated?

A hint that the answer to the question is not so obvious comes from the fact that the answer is *negative* if "semidegrees" in the question are replaced by "degree-like functions". Indeed, [3] presents two finitely generated subrings A_1 and A_2 of the complex polynomial ring in 32 variables such that $A_1 \cap A_2$ is *not* finitely generated over $\mathbb{C}$ [*]. Therefore, if δ_1 and δ_2 are any two finitely generated degree-like functions on $\mathbb{C}[x_1, \ldots, x_{32}]$ such that $\{f : \delta_i(f) = 0\} = A_i$, $1 \leq i \leq 2$, then $\delta := \max\{\delta_1, \delta_2\}$ is not finitely generated. This example suggests the necessity of restrictions on the degree zero component of the graded ring. In the case that $\dim A = 2$, we give (in Theorem 6.9) a positive answer to question 1 under one such restriction - that the degree zero component of the graded ring corresponding to each semidegree is $\mathbb{K}$.

Invertibility of the matrix of linking numbers at infinity: Given a finitely generated subdegree δ on the coordinate ring of an affine variety X, we may form the corresponding *matrix L_δ of linking numbers at infinity.* More precisely, if the associated semidegrees of δ are $\delta_1, \ldots, \delta_N$, then L_δ is the $N \times N$ matrix with entries $l_{ij} :=$ the linking number at infinity of δ_i and δ_j, $1 \leq i, j \leq N$. The diagonal entries of L_δ are all 1's and $l_{ij} l_{ji} > 1$ for all $i \neq j$,

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[*] I have learned about this fact from the answer to a question posted on *http://mathoverflow.net* [1]

$1 \le i, j \le N$. A simple consequence of this observation is that if $N = 2$, then L_δ is invertible. This motivates us to ask the following question:

Question 2. Is L_δ invertible for all δ?

When $\dim X = 2$ and δ_j's are 'complete', i.e. they satisfy the hypothesis of our generalization of Gordan's lemma, we show (in Proposition 6.12) that the answer to question 2 is positive. A somewhat curious (and easy to prove directly) consequence (which follows from taking $X = \mathbb{K}^2$ and δ_j's to be weighted degrees) is the following:

> Let $k \ge 1$ and $v_1, \ldots, v_k$ be mutually non-proportional elements of $\mathbb{Q}^2$ with positive coordinates. Let L be the $k \times k$ matrix with entries $l_{ij} := \max\{v_{ik}/v_{jk} : 1 \le k \le 2\}$. Then L is invertible. $(*)$

Our final question is about a natural generalization of $(*)$:

Question 3. Does $(*)$ remain true with $\mathbb{Q}^2$ being replaced by $\mathbb{Q}^m$ for all $m \ge 1$?

In fact, a positive answer to question 3 is equivalent to a positive answer to question 2 for complete semidegrees and all dimensions. We think that both of these questions have positive answers in much more general setting than we prove here, but as we explain in section 6, our proofs break down in dimensions larger than 2.

1.3 Notations and organization

Throughout this article, unless stated otherwise, $\mathbb{K}$ is an algebraically closed field, X is an irreducible affine algebraic variety over $\mathbb{K}$, A is the coordinate ring of X, $\mathcal{F}$ is a *filtration* on A and δ is a degree-like function on A. For a Cartier divisor D on a variety, we denote its corresponding Weil divisor by $[D]$.

In section 2 we introduce *filtrations* and give a characterization (in theorem 2.2) for the completions of affine varieties which are determined by filtrations on their coordinate rings. The first half of proposition 2.2 is well-known, but we include it for the sake of completion. After giving some classical examples of completions determined by filtrations, we present an example due to Hironaka of projective completions which do *not* come from filtrations. We also give an example of an embedding $X \overset{\psi}{\hookrightarrow} \bar{X} \subseteq \mathbb{P}^N(\mathbb{K})$ such that $\bar{X}$ is isomorphic to a projective completion of X determined by a filtration on A, but the embedding ψ does *not* come from any filtration.

We introduce degree-like functions corresponding to filtrations in section 3 and present examples of completions determined by semidegrees and subdegrees. In particular we show that normal projective toric varieties come from subdegrees on the ring of Laurent polynomials and give an example of an *iterated semidegree* - a notion which will be tackled in depth in the forthcoming continuation of this article. We also give an example which shows that the valuations corresponding to the semidegrees associated to a subdegree do *not* in general determine the completion corresponding to the subdegree.

In section 4 we establish the basic structure of subdegrees. Our first theorem (Theorem 4.1) classifies the filtrations determined by semi- and subdegrees. As a corollary we deduce that for a completion $\psi : X \hookrightarrow X^\delta$ given by a subdegree δ, the irreducible components of $X_\infty := X^\delta \setminus X$ are in a one-to-one correspondence with the unique minimal collection of non-trivial semidegrees defining δ. The main result of subsection 4.2 is a technical result (namely Theorem 4.7) which is fundamental for later development. More specifically, Theorem 4.7 states that for a *finitely generated* subdegree, its associated semidegrees are *integer valued* (i.e. they do not take $-\infty$ as a value).

Subsection 5.1 is devoted to the study of 'normality' properties of a completion X^δ of X determined by a subdegree on A. Proposition 5.1 shows that X^δ is *nonsingular in codimension one at infinity*, i.e. for each irreducible component V of X_∞, the local ring $\mathcal{O}_{V,X^\delta}$ of X^δ along V is regular and hence is a discrete valuation ring; moreover, the semidegree associated to V is proportional to the valuation associated to $\mathcal{O}_{V,X^\delta}$. We then introduce the notion of *relative normality at infinity*, and show (in Proposition 5.7) that X^δ is relatively normal at infinity with respect to X. We also introduce the notion of the *normalization at infinity* of a variety Y with respect to a dense Zariski open subset U. Given an arbitrary completion $X \hookrightarrow Z$ which comes from a filtration, we show in our Main Existence Theorem (Theorem 5.12) that there is a subdegree δ such that the corresponding completion X^δ of X is the normalization at infinity of Z with respect to X. Finally, for an arbitrary X and a subdegree δ on A, we describe in Proposition 5.15 the normalization of X^δ in terms of δ and the normalization of X. For an arbitrary degree-like function δ, in subsection 5.2 we introduce a canonical *divisor at infinity* on X^δ and prove a formula for the 'infinite part' of the pull-back of the divisor at infinity under a dominant morphism between completions of affine varieties determined by subdegrees.

In section 6, we take a closer look at the case of surfaces. In Proposition 6.5 we show that the completions of affine surfaces corresponding to subdegrees are uniquely determined by the orders of vanishing along the components of the curves at infinity (recall that this is *not* true in dimension ≥ 3: see Example 3.5). The main result of this section is Theorem 6.9 which states that if $\dim X = 2$ and all of the associated semidegrees of a subdegree δ on A are *complete* (i.e. the graded ring corresponding to each semidegree is a finitely generated algebra over $\mathbb{K}$ and degree-zero component of the graded ring is precisely $\mathbb{K}$), then δ is also finitely generated. As noted earlier, this is an analogue of *Gordan's lemma* in toric geometry. Given a finite collection of (mutually non-proportional) complete semidegrees on A, we show in Proposition 6.12 that corresponding *matrix of linking numbers at infinity* is invertible. As an application we compute the Picard group of X^δ in the case that X is an affine surface with trivial Picard group.

This article and its sequel are based on my PhD thesis. I express my gratitude to my advisor Professor Pierre Milman for posing the questions, helpful suggestions, and in general guiding me throughout this work. I would also like to thank Professor Khovanskii for helpful suggestions (e.g. he pointed out the possibility of a connection between semidegrees and 'orders of poles at infinity' [12]) and Professor Bernard Teissier for bringing the article [21] to my attention.

2 Filtrations

Throughout this section X will be an (irreducible) affine algebraic variety over $\mathbb{K}$ and A will be the coordinate ring of X.

Definition. A filtration $\mathcal{F}$ on A is a family $\{F_i : \ i \in \mathbb{Z}\}$ of $\mathbb{K}$-vector subspaces of A such that

1. $F_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$
2. $1 \in F_0 \setminus F_{-1}$
3. $A = \bigcup_{i \in \mathbb{Z}} F_i$ and
4. $F_i F_j \subseteq F_{i+j}$ for all $i, j \in \mathbb{Z}$.

Remark. We introduce the condition $1 \notin F_{-1}$ in order to exclude the trivial filtration, i.e. filtration $\{F_i := A\}_{i \in \mathbb{Z}}$.

Filtration $\mathcal{F}$ is called *non-negative* if $F_i = 0$ for all $i < 0$. Associated to each filtration $\mathcal{F}$ there are two graded rings:

$$A^{\mathcal{F}} := \bigoplus_{i \in \mathbb{Z}} F_i \quad \text{and}$$
$$\operatorname{gr} A^{\mathcal{F}} := \bigoplus_{i \in \mathbb{Z}} (F_i / F_{i-1}).$$

We denote a copy of $f \in F_d$ in the d-th graded component of $A^{\mathcal{F}}$ by $(f)_d$. $A^{\mathcal{F}}$ is given the structure of a graded $\mathbb{K}$-algebra with multiplication defined by:

$$(\sum_d (f_d)_d)(\sum_e (g_e)_e) := \sum_k \sum_{d+e=k} (f_d g_e)_k.$$

$\mathcal{F}$ is called a *finitely generated* filtration if $A^{\mathcal{F}}$ is a finitely generated $\mathbb{K}$-algebra.

Let t be an indeterminate over A. Then there is an isomorphism

$$A^{\mathcal{F}} \cong \sum_{i \in \mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}] \tag{1}$$

which maps $(1)_1 \mapsto t$. The following property of $A^{\mathcal{F}}$ is a straightforward corollary of this isomorphism.

Lemma 2.1. *$A^{\mathcal{F}}$ is a domain if and only if A is a domain.* □

If $\mathcal{F}$ is non-negative, finitely generated and $F_0 = \mathbb{K}$, then $X^{\mathcal{F}} := \operatorname{Proj} A^{\mathcal{F}}$ is projective, and hence is a complete variety. In view of this fact we make the following

Definition. A filtration $\mathcal{F} = \{F_d\}_{d \in \mathbb{Z}}$ on A is called *complete* if it is non-negative, finitely generated and $F_0 = \mathbb{K}$.

We will establish below (in Proposition 2.2) the relation between filtrations on A and projective completions of X. At first we recall some notations and facts which are used frequenlty in this article.

The d-th truncated subring of a graded ring $S = \bigoplus_{k\in\mathbb{Z}} S_k$ is $S^{[d]} := \bigoplus_{k\in\mathbb{Z}} S_{kd}$. The inclusion $S^{[d]} \hookrightarrow S$ induces an isomorphism $\operatorname{Proj} S \cong \operatorname{Proj} S^{[d]}$. For a subvariety Z of $\operatorname{Proj} S$, the induced embedding of Z into $\operatorname{Proj} S^{[d]}$ is called the *d-uple embedding* of Z. If S is a finitely generated $\mathbb{K}$-algebra, then $S^{[d]}$ is also finitely generated as a $\mathbb{K}$-algebra and, choosing a set of generators of $S^{[d]}$, one can embed $\operatorname{Proj} S$ as a subvariety of an appropriate weighted projective space. Moreover, there exists d such that $S^{[d]}$ is generated as a $\mathbb{K}$-algebra by $(S^{[d]})_1 = S_d$ (e.g. it suffices to take $d = (k+1)d'$, where d' is the least common multiple of $d_0, \ldots, d_k$ [17, Lemma in section III.8]). In that case the d-uple embedding embeds $\operatorname{Proj} S$ into a usual projective space.

We also use heavily in this article the language of divisors and mostly follow the notations of [7]. In particular, for a codimension one subvariety Y of X, we write $[Y]$ for the corresponding Weil divisor. Let $D := \sum_{i=1}^{N} a_i[Y_i]$ be a Weil divisor on X. We say that D is *effective* and write $D \geq 0$ if $a_i \geq 0$ for all i, $1 \leq i \leq N$. The *support* of D is $\operatorname{Supp} D := \cup\{Y_i : a_i \neq 0\}$. Also, recall that if D is a Cartier divisor on X, then D is called *very ample* if there exists a closed immersion $\iota : \bar{X} \hookrightarrow \mathbb{P}^N(\mathbb{K})$ such that D is the pullback via ι of a hyperplane section of $\mathbb{P}^N(\mathbb{K})$. Finally, D is called *ample* if there exists a positive integer k such that kD is very ample.

Proposition 2.2. *If $\mathcal{F}$ is a complete filtration on $A := \mathbb{K}[X]$, then there is an open immersion $\psi_{\mathcal{F}}$ of X onto a dense open subvariety of $X^{\mathcal{F}}$. The complement X_∞ of X in $X^{\mathcal{F}}$ is isomorphic to* Proj gr $A^{\mathcal{F}}$ *and is the support of an effective ample Cartier divisor on $X^{\mathcal{F}}$. In partcicular, $X_\infty = V((1)_1) \cong$* Proj gr $A^{\mathcal{F}}$*. Conversely, given any projective completion $\phi : X \hookrightarrow \bar{X}$ of X such that $\bar{X} \setminus X$ is the support of an effective ample Cartier divisor on $\bar{X}$, there is a complete filtration $\mathcal{F}$ on A and an isomorphism $\Phi : X^{\mathcal{F}} \to \bar{X}$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & X \arrow[dl, "\psi_{\mathcal{F}}"'] \arrow[dr, "\phi"] & \\
X^{\mathcal{F}} \arrow[rr, "\Phi"] & & \bar{X}
\end{tikzcd}$$

Proof. Recall that basic open sets in $X^{\mathcal{F}} = \operatorname{Proj} A^{\mathcal{F}}$ are given by $D(G) := \{Q \in X^{\mathcal{F}} | G \notin Q\} = \operatorname{Spec}(A^{\mathcal{F}})_{(G)}$, where G ranges over the homogeneous elements in $A^{\mathcal{F}}$, and $(A^{\mathcal{F}})_{(G)}$ is the subring of $(A^{\mathcal{F}})_G := A^{\mathcal{F}}[\frac{1}{G}]$ consisting of degree zero homogeneous elements. Identify $A^{\mathcal{F}}$ with $\sum_{i\in\mathbb{Z}} F_i t^i$ via (1). Then $(A^{\mathcal{F}})_{(t)} = \{gt^d/t^d : g \in F_d \subseteq A,\ d \geq 0\} \cong \{g : g \in F_d \subseteq A\} = A$ as a subring of $A[t, t^{-1}]$; hence $X = \operatorname{Spec} A \cong \operatorname{Spec}(A^{\mathcal{F}})_{(t)} \cong D(t)$ and $X^{\mathcal{F}} \setminus X = V(t)$. Since $A^{\mathcal{F}}$ is a domain (lemma 2.1), it follows that X is dense in $X^{\mathcal{F}}$. Now, let $\pi : A^{\mathcal{F}} \to \operatorname{gr} A^{\mathcal{F}}$ be the natural projection. It is straightforward to see that π is a surjective homomorphism of graded rings with kernel $\langle t \rangle$ (the ideal generated by t in $A^{\mathcal{F}}$). It follows that $\operatorname{Proj}(\operatorname{gr} A^{\mathcal{F}}) \overset{\pi^*}{\cong} V(t)$. Finally, choosing d such that the d-uple embedding embeds $X^{\mathcal{F}}$ into a usual projective space $\mathbb{P}^N(\mathbb{K})$, we see that $V(t) = V(t^d)$ is the support of (the pullback of) a hyperplane section of $\mathbb{P}^N(\mathbb{K})$, which is by definition a very ample and effective divisor. This completes the proof of the first assertion.

Now let $\phi : X \hookrightarrow \bar{X}$ be a projective completion of X such that $\bar{X} \setminus X$ is the support of an effective ample Cartier divisor D on $\bar{X}$. Replacing D by kD for sufficiently large k, we may assume that there exists a closed immersion $\iota : \bar{X} \hookrightarrow \mathbb{P}^N(\mathbb{K})$ such that D is the pullback via ι of a hyperplane section of $\mathbb{P}^N(\mathbb{K})$. Let $S := \mathbb{K}[x_0, \ldots, x_N]/I$ be the *homogeneous coordinate ring* of $\bar{X}$. Let $a_0, \ldots, a_N \in \mathbb{K}$ be such that $\operatorname{Supp} D = V(H) \cap \bar{X}$, where $H := \sum_{i=0}^N a_i x_i$. Then $X \overset{\iota\circ\phi}{\cong} D(h) \subseteq \operatorname{Proj} S$, where h is the equivalence class of H in S, and the isomorphism of varieties $D(h) \cong \operatorname{Spec} S_{(h)}$ induces an isomorphism of rings $\phi^* : S_{(h)} \cong A$. It follows that for each $g \in A$, $(\phi^*)^{-1}(g) = a/h^k$ for some $k \geq 0$ and $a \in S_k$. Define

$$F_k := \phi^*(S_k/h^k) = \{g \in R | \ (\phi^*)^{-1}(g) \in S_k/h^k\}.$$

Then it is easy to see that $\mathcal{F} = \{F_i\}_{i \geq 0}$ is a filtration on A. By means of this filtration we construct, as usual, the ring $A^{\mathcal{F}} := \bigoplus_{d \geq 0} F_d$. Map $\phi^* : S_{(h)} \to A$ induces a map $\Phi^* : S \to A^{\mathcal{F}}$ which sends each $g \in S_d$ to $(\phi^*(g/h^d))_d$, i.e. to the copy of $\phi^*(g/h^d)$ in $F_d \subseteq A^{\mathcal{F}}$. Consequently Φ^* is a surjective homomomorphism of graded rings. Moreover, irreducibility of $\bar{X}$ implies that $\ker \Phi^* = 0$, so that the induced map $\Phi : \operatorname{Proj} A^{\mathcal{F}} \to \operatorname{Proj} S \cong \bar{X}$ is an isomorphism.

To see that the isomorphism $\operatorname{Proj} A^{\mathcal{F}} \cong \operatorname{Proj} S$ is the identity map when restricted to $\operatorname{Spec} A$, note that maps $\phi : \operatorname{Spec} A \to \operatorname{Proj} S$ and $\psi_{\mathcal{F}} : \operatorname{Spec} A \to \operatorname{Proj} A^{\mathcal{F}}$ are completely determined by the corresponding pull back maps $\phi^* : S_{(f)} \to A$ and $\psi_{\mathcal{F}}^* : (A^{\mathcal{F}})_{(t)} \to A$ on the localizations of the respective graded $\mathbb{K}$-algebras, where as before $t := (1)_1 \in A^{\mathcal{F}}$. The latter two maps give rise to the following commutative diagrams:

$$\begin{tikzcd}
A \arrow[r, "1_A"] & A \\
S_{(f)} \arrow[u, "\phi^*"] \arrow[r, "\Phi^*"] & (A^{\mathcal{F}})_{(t)} \arrow[u, "\psi_{\mathcal{F}}^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi^*(g/f^d) \arrow[r, mapsto, "1_A"] & \phi^*(g/f^d) \\
g/f^d \arrow[u, mapsto, "\phi^*"] \arrow[r, mapsto, "\Phi^*"] & \frac{(\phi^*(g/f^d))_d}{t^d} \arrow[u, mapsto, "\psi_{\mathcal{F}}^*"]
\end{tikzcd}$$

Since the top horizontal map and both of the vertical maps are isomorphisms, it follows that the bottom horizontal map is an isomorphism as well, which concludes the proof of proposition 2.2. $\square$

Remark 2.3. 1. If $\mathcal{F}$ is non-negative and finitely generated (but not necessarily complete), then $X^{\mathcal{F}}$ is a quasi-projective variety (over $\mathbb{K}$) and the arguments in the first part of the proof of proposition 2.2 go through to show that there is an open immersion of X onto a dense open subvariety of $X^{\mathcal{F}}$ and $X_\infty := X^{\mathcal{F}} \setminus X = V((1)_1) \cong \operatorname{Proj} \operatorname{gr} A^{\mathcal{F}}$.

2. Similarly, the arguments in the second part of the proof show that if $S = \bigoplus_{d \in \mathbb{Z}} S_d$ is a graded $\mathbb{K}$-algber and $\phi : X \hookrightarrow \operatorname{Proj} S$ is an open immersion of X onto a Zariski dense subset of $\operatorname{Proj} S$ such that $\operatorname{Proj} S \setminus \operatorname{Spec} A = V(F)$ for some $F \in S_d$, $d \geq 1$, then there is a filtration $\mathcal{F}$ on A such that $S^{[d]} \cong A^{\mathcal{F}}$ and the induced isomorphism $\Phi : X^{\mathcal{F}} \hookrightarrow \operatorname{Proj} S$ restricts to ϕ on X.

Now we work out some examples of complete filtrations on the coordinate rings of affine varieties, and determine the corresponding completions. For the first three examples we set $X = \mathbb{K}^n$ and $A = \mathbb{K}[x_1, \ldots, x_n]$.

Example 2.4. Let $d_1, \ldots, d_n$ be any n positive integers and F_d be the $\mathbb{K}$-linear span of all the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ such that $\sum \alpha_i d_i \leq d$. Set $d_0 := 1$. Then F_d can be identified with the set of *weighted homogeneous* polynomials in $x_0, \ldots, x_n$ of weighted degree

d, where weight of x_i is d_i for each i. It follows that $A^{\mathcal{F}}$ is isomorphic as a graded $\mathbb{K}$-algebra to $\mathbb{K}[x_0, \dots, x_n]$, where the grading on the latter is induced by the weighted degree $(d_0, d_1, \dots, d_n)$. Consequently, $X^{\mathcal{F}}$ is the weighted projective space $\mathbb{P}^n(\mathbb{K}; d_0, d_1, \dots, d_n)$.

Example 2.5. Let F_k be the set of polynomials of degree less than or equal to dk, where d is a fixed positive integer. Then $X^{\mathcal{F}}$ is the d-uple embedding of $\mathbb{P}^n(\mathbb{K})$ in $\mathbb{P}^{m-1}(\mathbb{K})$, where $m := \binom{n+d}{n}$ = number of all monomials in n variables of degree at most d. In particular, for $n = 1$, $X^{\mathcal{F}}$ is the rational canonical curve of degree d in $\mathbb{P}^d(\mathbb{K})$.

Example 2.6. X is again $\mathbb{K}^n$ as above. Assume $n \geq 2$. Fix an integer k with $1 \leq k < n$. Let F_1 be the $\mathbb{K}$-linear span of all monomials of degree less than or equal to two excluding those of the form $x_i x_j$ with $i \geq j > k$. Let $F_d = (F_1)^d$ for $d \geq 1$. Then $X^{\mathcal{F}}$ is isomorphic to the variety resulting from blowing up $\mathbb{P}^n(\mathbb{K})$ along the subspace $V := V(x_0, \dots, x_k)$.

Example 2.7. Let X be a normal affine variety with trivial Picard group $\operatorname{Pic} X$ (e.g. $\mathbb{K}^n$, $(\mathbb{K}^*)^n$, or any X whose coordinate ring is a unique factorization domain). Let $\bar{X} \subseteq \mathbb{P}^N(\mathbb{K})$ be a normal projective completion of X such that $X_\infty := \bar{X} \setminus X$ is irreducible. Then the embedding $X \hookrightarrow \bar{X}$ arises from a filtration. Indeed, let D be the Cartier divisor on $\bar{X}$ corresponding to a hyperplane section of $\mathbb{P}^N(\mathbb{K})$. The Weil divisor associated to D is of the form $[D] = a_0[X_\infty] + \sum_{j=1}^m a_j[\bar{V}_j]$, where for each j, $\bar{V}_j$ is the closure in $\bar{X}$ of a codimension one subvariety V_j of X. Since $\operatorname{Pic} X = 0$, there is a rational function f on X such that the principal divisor of f on X is $[\operatorname{div}_X(f)] = \sum_{j=1}^m a_j[V_j]$. Let $[D'] := [D] - [\operatorname{div}_{\bar{X}}(f)]$, where $[\operatorname{div}_{\bar{X}}(f)]$ is the principal divisor of f on $\bar{X}$. Then $[D'] = a'[X_\infty]$ for some $a' \in \mathbb{Z}$. Since $a' = \deg D' = \deg D = \deg \bar{X} > 0$ [10, Exercise II.6.2], and D' is *very ample*, it follows according to proposition 2.2 that there is a filtration $\mathcal{F}$ on X such that $X^{\mathcal{F}} \cong \bar{X}$.

Example 2.8. Not all projective completions of affine varieties are determined by a filtration, as the following example of Hironaka (see, e.g., [9]) shows. Let $\phi : V' \to V$ be a morphism of 3-dimensional projective varieties, with V' non-singular and V non-singular except at a point P, such that $\phi^{-1}(P)$ is a curve and ϕ is an isomorphism on the rest of V' and V. Let W be a hyperplane section of V through P. Then $X := V' \setminus \phi^{-1}(W) \cong V \setminus W$ is affine, but $\phi^{-1}(W)$ is *not* the support of any effective ample divisor on V' [9, Section 1]. Therefore proposition 2.2 implies that there is no filtration $\mathcal{F}$ on A such that $X^{\mathcal{F}} \cong V'$.

Example 2.9. A filtration $\mathcal{F}$ on A does not only gives an abstract projective completion of X, but (via choices of $\mathbb{K}$-algebra generators of $A^{\mathcal{F}}$) also embeddings of X into weighted projective spaces. As the following example (which is a variation of an example by Mike Roth and Ravi Vakil [18, Example 2.5(a)]) shows, even if a projective completion $\phi : X \hookrightarrow \bar{X} \subseteq \mathbb{P}^N(\mathbb{K})$ is isomorphic to $X^{\mathcal{F}}$ for some filtration $\mathcal{F}$ on A, it is *not* necessarily true that ϕ is induced by (in the sense of the first sentence of this example) a set of $\mathbb{K}$-algebra generators of $A^{\mathcal{F}}$. Let X' be a nonsingular cubic curve in $\mathbb{P}^2(\mathbb{K})$. Let O be one of its 9 inflection points. Recall that X' can be given the structure of an abelian group with O as the origin and that $\phi : P \to [P] - [O]$ gives an injective group homomorphism of X' into its class group $\operatorname{Cl} X'$ [10, Example II.6.10.2]. There are only countably many points $P \in X'$ such that P is a torsion point, i.e. $k \cdot P = 0$ for some $k > 0$ [20, Section III.3.4]. Pick any *non-torsion* point P of X'. Then $\{P\}$ is the support of an ample divisor on X' and $X := X' \setminus \{P\}$ is an affine variety [9, Proposition 5]. Therefore, according to proposition 2.2, there exists a filtration $\mathcal{F}$ on A such that $X^{\mathcal{F}} \cong X'$. On the other hand, one can show that there is no homogeneous polynomial f in $\mathbb{K}[x_0, x_1, x_2]$ such that $V(f) \cap X' = \{P\}$ [15, Remark 1]. It follows due to the first assertion of proposition

2.2 that there is no integer $d > 0$ such that the embedding of X into the image of the d-uple embedding of X' is induced by a filtration!

3 Semidegree and Subdegree

Definition. A *degree-like function* on A is a map $\delta : A \setminus \{0\} \to \mathbb{Z} \cup \{-\infty\}$ such that:

1. $\delta(\mathbb{K}) = 0$.
2. $\delta(f + g) \leq \max\{\delta(f), \delta(g)\}$ for all $f, g \in A$, with $<$ in the preceding equation implying $\delta(f) = \delta(g)$.
3. $\delta(fg) \leq \delta(f) + \delta(g)$ for all $f, g \in A$.

Remark. Even though we allow degree-like functions to have values $-\infty$, it will follow from theorems 4.7 and 5.12 that we do not need to leave the realm of integer valued degree-like functions by either of the operations of *normalizing* degree-like functions or *taking associated semidegrees of a subdegree* (we introduce both notions below). Intermediately we allow for a theoretical possibility of ending up with a degree-like function that takes the value $-\infty$ on some nonzero $f \in A$.

There is a one-to-one correspondence between degree-like functions and filtrations:

$$\begin{array}{ccc} \textit{Filtrations} & \longleftrightarrow & \textit{degree-like functions} \\ \mathcal{F} = \{F_d\}_{d\in\mathbb{Z}} & \longrightarrow & \delta_{\mathcal{F}} : f \in A \mapsto \inf\{d : f \in F_d\} \\ \mathcal{F}_\delta := \{F_d := \{f \in A : \delta(f) \leq d\}\}_{d\in\mathbb{Z}} & \longleftarrow & \delta \end{array}$$

In the remainder of this article we identify degree-like functions with the corresponding filtrations. In particular, we refer to a degree-like function δ as *complete* (resp. *finitely generated*) iff the corresponding filtration $\mathcal{F}_\delta$ is complete (resp. finitely generated). Moreover, A^δ and $\operatorname{gr} A^\delta$ will be shorthand notations for the rings $A^{\mathcal{F}_\delta}$ and, respectively, $\operatorname{gr} A^{\mathcal{F}_\delta}$ and ψ_δ will denote the natural embedding $\operatorname{Spec} A \hookrightarrow X^\delta := \operatorname{Proj} A^\delta$, while for the sake of convenience we would freely refer to X^δ as the "completion ψ_δ" (of $\operatorname{Spec} A$).

The protagonists of this article are two classes of degree-like functions which satisfy stronger versions of the multiplicative property (i.e. property 3 above).

Definition.

- A degree-like function δ on A is a *semidegree* iff $\delta(fg) = \delta(f)+\delta(g)$ for all $f, g \in A\setminus\{0\}$.
- We say that δ is a *subdegree* if there are semidegrees $\delta_1, \ldots, \delta_N$ such that

$$\delta(f) = \max_{1\leq i\leq N} \delta_i(f) \quad \text{for all } f \in A \setminus \{0\}. \tag{2}$$

Given a subdegree δ as in (2), we may assume by getting rid of some δ_i's, if need be, that every δ_i that appears in (2) is *not redundant* in the sense that for every i, there is an $f \in A$ such that $\delta_i(f) > \delta_j(f)$ for all $j \neq i$ (indeed, if there is an i such that for all $f \in A$, $\delta_i(f) \leq \delta_j(f)$ for some $j \neq i$, then $\delta(f) = \max_{j\neq i} \delta_j(f)$ for all $f \in A$). In the latter case we will say that (2) is a *minimal presentation* of δ.

Remark 3.1. Note that the negative of a semidegree is a *discrete valuation.* The negative of a degree-like function is usually called a (discrete) *pseudo-valuation* and the negative of a subdegree is sometimes called a (discrete) *subvaluation* (see e.g. [11]).

Example 3.2 (Weighted Degree)**.** Every weighted degree δ on the polynomial ring $A := \mathbb{K}[x_1, \ldots, x_n]$ is a semidegree.

Example 3.3 (An iterated semidegree)**.** Let $X := \mathbb{K}^2$ and $A := \mathbb{K}[x_1, x_2]$. Define a filtration $\mathcal{F} := \{F_d : d \geq 0\}$ on A by setting $F_0 := \mathbb{K}$, $F_1 := \mathbb{K}\langle 1, x_1^2 - x_2^3\rangle$, $F_2 := (F_1)^2 + \mathbb{K}\langle x_2\rangle$, $F_3 := F_1F_2 + \mathbb{K}\langle x_1\rangle$ and $F_d := \sum_{j=1}^{d-1} F_jF_{d-j}$ for $d \geq 4$. We claim that function $\delta := \delta_{\mathcal{F}}$ is a semidegree. Indeed,

$$A^\delta = \mathbb{K}[(1)_1, (x_1)_3, (x_2)_2, (x_1^2 - x_2^3)_1] \cong \mathbb{K}[X_1, X_2, Y, Z]/\langle YZ^5 - X_1^2 + X_2^3\rangle ,$$

where the last isomorphism is induced by a $\mathbb{K}$-algebra homomorphism which sends $X_1 \mapsto (x_1)_3$, $X_2 \mapsto (x_2)_2$, $Y \mapsto (x_1^2 - x_2^3)_1$ and $Z \mapsto (1)_1$. The inverse image of the ideal $I := \langle (1)_1\rangle$ of A^δ under this isomorphism coincides with ideal $\langle Z, YZ^5 - X_1^2 + X_2^3\rangle = \langle Z, X_1^2 - X_2^3\rangle$. Since the latter is a prime ideal of $\mathbb{K}[X_1, X_2, Y, Z]$, it follows that I is a prime ideal of A^δ as well. Theorem 4.1 then implies that δ is a semidegree. The isomorphism constructed above induces a closed embedding of X^δ onto hypersurface $V(YZ^5 - X_1^2 + X_2^3) \subseteq \mathbf{W}\mathbb{P}$, where $\mathbf{W}\mathbb{P}$ is the weighted projective space $\mathbb{P}^3(\mathbb{K}; 1, 3, 2, 1)$ with (weighted homogeneous) coordinates $[Z : X_1 : X_2 : Y]$. The semidegree δ is an example of an *iterated semidegree* (to be introduced in the forthcoming continuation of this article).

Example 3.4 (Subdegrees determined by rational polytopes)**.** Let X be the n-torus $(\mathbb{K}^*)^n$ and $A := \mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be its coordinate ring. Let $\mathcal{P}$ be a convex rational polytope (i.e. a convex polytope in $\mathbb{R}^n$ with vertices in $\mathbb{Q}^n$) of dimension n containing origin in its interior. Define a function $\delta'_{\mathcal{P}} : A \setminus \{0\} \to \mathbb{Q}_+$ as follows:

$$\begin{aligned}\delta'_{\mathcal{P}}(x^\alpha) &:= \inf\{r \in \mathbb{Q}_+ : \alpha \in r\mathcal{P}\}, \text{ and}\\ \delta'_{\mathcal{P}}(\sum a_\alpha x^\alpha) &:= \max_{a_\alpha \neq 0} \delta'_{\mathcal{P}}(x^\alpha).\end{aligned}$$

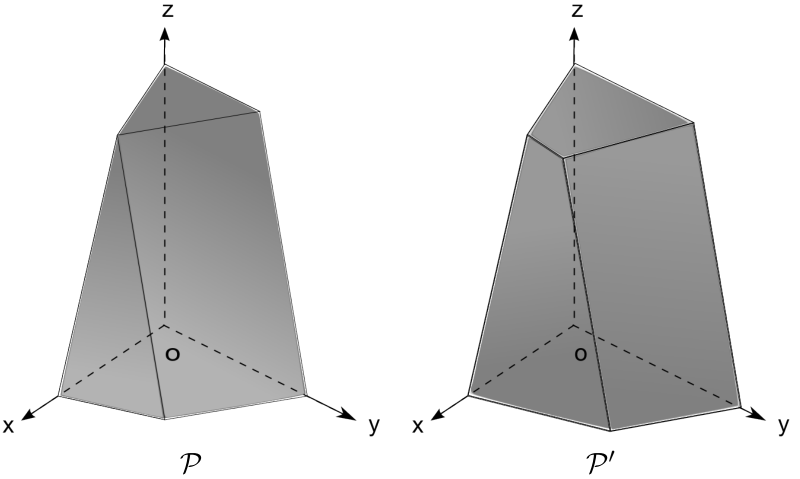
For each facet $\mathcal{Q}$ of $\mathcal{P}$, let $\omega_{\mathcal{Q}}$ be the smallest 'outward pointing' integral vector normal to $\mathcal{Q}$ and let $c_{\mathcal{Q}} = \langle \omega_{\mathcal{Q}}, \alpha\rangle$, where α is any element of the hyperplane that contains $\mathcal{Q}$. Since $\mathcal{P}$ is rational, in each $\mathcal{Q}$ there is an α with rational coordinates, and therefore each $c_{\mathcal{Q}}$ is a positive rational number. Let $\delta'_{\mathcal{Q}}$ be the $\mathbb{Q}$-valued weighted degree on A given by:

$$\delta'_{\mathcal{Q}}(\sum a_\alpha x^\alpha) := \max_{a_\alpha \neq 0} \frac{\langle \omega_{\mathcal{Q}}, \alpha\rangle}{c_{\mathcal{Q}}}.$$

It is straightforward to see that for each $\alpha \in \mathbb{R}^n$, $\delta'_{\mathcal{P}}(x^\alpha) = \max\{\delta'_{\mathcal{Q}}(x^\alpha) : \mathcal{Q}$ is a facet of $\mathcal{P}\}$. Let $k \in \mathbb{N}$ be such that $k/c_{\mathcal{Q}}$ is an integer for each $\mathcal{Q}$. Then $k\delta'_{\mathcal{Q}}$ is an integer valued semidegree for each $\mathcal{Q}$ and hence $k\delta'_{\mathcal{P}}$ is a subdegree.

It follows via Gordan's lemma [6, Proposition 1, Section 1.2] that $A^{k\delta'_{\mathcal{P}}}$ is finitely generated. Let $\delta_{\mathcal{P}} := k\delta'_{\mathcal{P}}$, where k is an integer as in the above claim. Let d be a positive integer such that $(A^{\delta_{\mathcal{P}}})^{[d]}$ is generated as a $\mathbb{K}$-algebra by elements in the d-th graded component of $A^{\delta_{\mathcal{P}}}$, or equivalently the $\mathbb{K}$-span of monomials x^α such that $\alpha \in \frac{d}{k}\mathcal{P}$. Then the image of the d-uple embedding of $X^{\delta_{\mathcal{P}}}$ is the closure (in the appropriate dimensional projective space) of the image of $(\mathbb{K}^*)^n$ under the map whose components are the monomials with exponents in $\frac{d}{k}\mathcal{P}$. It follows that $X^{\delta_{\mathcal{P}}}$ is isomorphic via the d-uple embedding to the *toric variety* $X_{\mathcal{P}}$ determined by $\mathcal{P}$ (see, e.g. [6, Section 3.4]).

Example 3.5 (Semidegrees contain more information than valuations)**.** Let δ be a complete subdegree on A with a minimal presentation $\delta = \max_{i=1}^N\{\delta_i\}$. Proposition 5.1 shows that the δ_i's are proportional to the orders of vanishing along the components of the hypersurface at infinity of X^δ. It is perhaps surprising that the orders of vanishing along the components of the hypersurface at infinity do *not* in general determine X^δ. We now give an example from toric geometry where this can be seen with $X = \mathbb{K}^3$ and δ_j's are weighted degrees. Note that 3 is the smallest dimension for which this is possible, i.e. in lower dimensions, for every subdegree δ, the orders of vanishing along the components of the hypersurface at infinity do uniquely determine X^δ (in dimension 1 this is true because the number of points of X_∞ is precisely the number of the components of the hypersurface at infinity, and X^δ is non-singular at each of those points (proposition 5.1); in dimension 2 this is precisely proposition 6.5).



Let δ_i, $1 \leq i \leq 3$, be weighted degrees on $\mathbb{K}[x_1, x_2, x_3]$ with weight vectors respectively $(3,3,2)$, $(1,1,1)$ and $(6,3,2)$. Define $\delta := \max\{4\delta_1, 9\delta_2, 3\delta_3\}$ and $\delta' := \max\{6\delta_1, 15\delta_2, 5\delta_3\}$. Then the hypersurfaces at infinity of both X^δ and $X^{\delta'}$ have 3 components and the orders of vanishing along these components in both cases are $\{-\delta_i : 1 \leq i \leq 3\}$ (Proposition 5.1). But the polytopes $\mathcal{P}$ and $\mathcal{P}'$ associated respectively to δ and δ' are combinatorially different: the intersection of the facets associated to δ_2 and δ_3 is a point in the former case, whereas it is a line segment in the latter. It follows that the intersection of the components (of the hypersurface at ∞) associated to δ_2 and δ_3 is a point in X^δ and a curve in $X^{\delta'}$. In particular, X^δ and $X^{\delta'}$ are *not* isomorphic.

4 Structure of Subdegrees

4.1 Uniqueness of minimal presentations of subdegrees by semidegrees

We start with the structure theorem for completions determined by subdegrees. Recall that an ideal I of a ring R is *decomposable* if it is the intersection of finitely many primary ideals of R.

Theorem 4.1 (Structure theorem for subdegrees)**.** *Let δ be a degree like function on A and I be the ideal of A^δ generated by $(1)_1$.*

1. δ is a semidegree if and only if I is a prime ideal.

2. δ is a subdegree if and only if I is a decomposable radical ideal.
3. If δ is a subdegree, then semidegrees $\delta_1, \ldots, \delta_N$ of a minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$ of δ are unique.
4. If A^δ is Noetherian, then δ is a subdegree if and only if I is a radical ideal.

Corollary 4.2. *Assume A^δ is Noetherian. Then δ is a subdegree on A iff $\delta(f^k) = k\delta(f)$ for all $f \in A$ and $k \geq 0$.*

Proof. This claim is a straightforward consequence of assertion 4 of theorem 4.1 and the observation that the ideal I of theorem 2 is radical iff $\delta(f^k) = k\delta(f)$ for all $f \in A$ and $k \geq 0$. ☐

The heart of the proof of theorem 4.1 lies in the proof of assertion 3. The other three assertions follow also from the following result of Szpiro [21]. Therefore we only give here the proof of assertion 3.[†]

Definition. Let ν be a discrete pseudo-valuation (see remark 3.1) on A. Then $-\nu$ is a degree-like function on A. The pseudo-valuation ν is called *homogeneous* if the graded ring $\operatorname{gr} A^{-\nu}$ has no nilpotents.

Theorem 4.3 (Szpiro [21, Theorem 1 and Proposition 3])**.** *Let ν be a discrete homogeneous pseudo-valuation on A. Then there exists a family $\{\nu_i\}_{i \in \mathcal{I}}$ of discrete valuations on A such that $\nu = \inf_{i \in \mathcal{I}} \nu_i$. It is possible to choose $\mathcal{I}$ such that $|\mathcal{I}|$ is the number of minimal ideals of $\operatorname{gr} A^{-\nu}$.* ☐

Proof of assertion 3 of theorem 4.1. Let $\delta = \max_{1 \leq i \leq N} \delta_i$ be a minimal presentation of δ. For each i with $0 \leq i \leq N$, fix an $f_i \in A$ such that

$$d_i := \delta_i(f_i) > \delta_j(f_i) \text{ for } 1 \leq j \neq i \leq N. \tag{3}$$

Then, in particular, $\delta(f_i) = d_i \in \mathbb{Z}$.

Lemma 4.1.1.
(1) For each i, $1 \leq i \leq N$, $(I : (f_i)_{d_i})$ is a homogeneous ideal, and the homogeneous elements of $(I : (f_i)_{d_i})$ are precisely the elements of $L_i := \{(f)_d : f \in A,\ d > \delta_i(f)\}$.
(2) For each i, $(I : (f_i)_{d_i})$ is a distinct minimal prime ideal of A^δ containing I. In particular, $(f_i)_{d_i} \in (\bigcap_{j \neq i} (I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i})$ for each i. Moreover, $\bigcap_{i=1}^N (I : (f_i)_{d_i}) = I$.

Proof. We first prove assertion 1. Fix an i, $1 \leq i \leq N$. Since $(f_i)_{d_i}$ is a homogeneous element in A^δ and I is a homogeneous ideal of A^δ, it follows that $(I : (f_i)_{d_i})$ is also a homogeneous ideal of A^δ. Let $(f)_d$ be an arbitrary homogeneous element of $(I : (f_i)_{d_i})$. If $\delta(f) < d$, then $\delta_i(f) \leq \delta(f) < d$ and $(f)_d \in L_i$. So assume $\delta(f) = d$. Since $(f)_d (f_i)_{d_i} = (ff_i)_{d+d_i} \in I$, it follows that $\delta(ff_i) < d + d_i$. But then $\delta_i(ff_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$, which implies that $\delta_i(f) < d = \delta(f)$, and thus $(f)_d \in L_i$. To summarize, all homogeneous elements of $(I : (f_i)_{d_i})$ belong to L_i.

Now let $(f)_d$ be an arbitrary element of L_i. If $d > \delta(f)$, then $(f)_d \in I \subseteq (I : (f_i)_{d_i})$. So assume $d = \delta(f)$. Then $\delta_i(f) < d = \delta(f)$, and thus $\delta_i(ff_i) = \delta_i(f) + \delta_i(f_i) < d + d_i$. Also, for

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[†] Theorem 4.1 was proved in all details in my PhD thesis [16, Section 2.2]. Professor Bernard Teissier later brought the article [21] to my attention - I thank him for this.

each $j \neq i$, $\delta_j(f_i) < d_i$, so that $\delta_j(ff_i) = \delta_j(f)+\delta_j(f_i) < d+d_i$. It follows that $\delta(ff_i) < d+d_i$ and $(f)_d(f_i)_{d_i} = (ff_i)_{d+d_i} \in I$, i.e. $(f)_d \in (I : (f_i)_{d_i})$, which proves inclusion $L_i \subseteq (I : (f_i)_{d_i})$ and completes the proof of assertion 1.

We now show that $(I : (f_i)_{d_i})$ is prime. Let $(g_1)_{e_1}, (g_2)_{e_2}$ be homogeneous elements of A^δ such that $(g_1)_{e_1}(g_2)_{e_2} \in (I : (f_i)_{d_i})$. Due to assertion 1, $(g_1g_2)_{e_1+e_2} = (g_1)_{e_1}(g_2)_{e_2} \in L_i$, which means that $\delta_i(g_1g_2) = \delta_i(g_1) + \delta_i(g_2) < e_1 + e_2$. Since $e_j \geq \delta(g_j) \geq \delta_i(g_j)$ for each j, it follows that there is j, $j = 1$ or 2, such that $\delta_i(g_j) < e_j$. Then assertion 1 implies that $(g_j)_{e_j} \in (I : (f_i)_{d_i})$, i.e. $(I : (f_i)_{d_i})$ is a prime ideal.

Next we show that $(I : (f_i)_{d_i})$ is a minimal prime ideal of A^δ containing ideal I. Indeed, if $I \subseteq \mathfrak{p} \subseteq (I : (f_i)_{d_i})$ for a prime ideal p of A^δ, and some $i \leq N$, then $(f_i)_{d_i} \notin \mathfrak{p}$ (since assertion 1 implies $(f_i)_{d_i} \notin (I : (f_i)_{d_i})$). But if $(g)_e \in (I : (f_i)_{d_i})$, then $(g)_e(f_i)_{d_i} \in I \subseteq \mathfrak{p}$ and it follows that $(g)_e \in \mathfrak{p}$. Hence $(I : (f_i)_{d_i}) \subseteq \mathfrak{p}$, and therefore $(I : (f_i)_{d_i}) = \mathfrak{p}$, as required.

To see that the ideals $(I : (f_i)_{d_i})$ are distinct, note that $(f_i)_{d_i} \in (\bigcap_{j\neq i} L_j) \setminus L_i$ for $1 \leq i \leq N$, which implies due to assertion 1 that $(f_i)_{d_i} \in (\bigcap_{j\neq i}(I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i})$ for every i.

Finally, pick any homogeneous $(g)_e \in \bigcap(I : (f_i)_{d_i})$. Then by assertion 1, for each i, $\delta_i(g) < e$. Therefore $\delta(g) = \max_i \delta_i(g) < e$, and hence $(g)_e \in I$. It follows that $\bigcap(I : (f_i)_{d_i}) = I$, which concludes the proof of the lemma. □

Let $\delta = \max_{1\leq i\leq N'} \delta'_i$ be another minimal presentation of δ. Then

$$d'_i := \delta'_i(f'_i) > \delta'_j(f'_i) \text{ for } 1 \leq j \neq i \leq N'. \tag{3'}$$

there exist $f'_1, \dots, f'_{N'} \in A$ such that $d'_i := \delta'_i(f'_i) > \delta'_j(f'_i)$ for $1 \leq j \neq i \leq N'$. Then d'_i are integers, and according to lemma 4.1.1,

$$I = \bigcap_{i=1}^{N}(I : (f_i)_{d_i}) \quad \text{and} \quad I = \bigcap_{i=1}^{N'}(I : (f'_i)_{d'_i})$$

are *unique* minimal primary decompositions of I. The latter uniqueness implies that $N = N'$ and, after an appropriate re-indexing of δ'_i's, that ideals $(I : (f_i)_{d_i})$ and $(I : (f'_i)_{d'_i})$ coincide for all $i \leq N$.

Fix an i, $1 \leq i \leq N$. According to assertion 2 of lemma 4.1.1, elements

$$(f'_i)_{d'_i} \in (\bigcap_{j\neq i}(I : (f'_j)_{d'_j})) \setminus (I : (f'_i)_{d'_i}) = (\bigcap_{j\neq i}(I : (f_j)_{d_j})) \setminus (I : (f_i)_{d_i})\,.$$

Therefore assertion 1 of lemma 4.1.1 implies that $(f'_i)_{d'_i} \in (\bigcap_{j\neq i} L_j) \setminus L_i$, where L_j's are as in lemma 4.1.1. It follows that $\delta_i(f'_i) = \delta(f'_i) = \delta'_i(f'_i)$, and $\delta_j(f'_i) < \delta_i(f'_i)$ for all $j \neq i$. The latter implies property (3) with f'_i's replacing f_i's. Since f_i's were assumed to be arbitrary elements in A such that (3) is true, we may assume without loss of generality that $f_i = f'_i$ for each i. It follows that $\delta_i = \delta'_i$ for all i by making use of

Lemma 4.1.2. *If $f_1, \ldots, f_N \in A$ satisfy* (3)*, then*

$$\delta_i(f) = \lim_{k\to\infty} \delta((f_i)^k f) - \delta((f_i)^k) \tag{4}$$

for all $f \in A$ and all i, $1 \le i \le N$.

Proof. Fix an i, $1 \le i \le N$. Let us write $\tilde{\delta}_i(f) := \lim_{k\to\infty} \delta((f_i)^k f) - \delta((f_i)^k)$ for all $f \in A$. We first show that $\tilde{\delta}_i$ is well defined for each i. Indeed, fix an i, $1 \le i \le N$. Let $f \in A$ and $k \ge 1$. Since $(f_i)_{d_i} \notin I$ and I is radical (assertion 2 of lemma 4.1.1), it follows that $((f_i)_{d_i})^k = ((f_i)^k)_{kd_i} \notin I$, so that $\delta((f_i)^k) = kd_i$. Therefore,

$$\begin{aligned} \delta((f_i)^{k+1} f) - \delta((f_i)^{k+1}) &\le \delta((f_i)^k f) + \delta(f_i) - \delta((f_i)^{k+1}) \\ &= \delta((f_i)^k f) + d_i - (k+1)d_i \\ &= \delta((f_i)^k f) - kd_i \\ &= \delta((f_i)^k f) - \delta((f_i)^k). \end{aligned} \tag{5}$$

It follows that $\tilde{\delta}_i(f)$ is a well defined element in $\mathbb{Z} \cup \{-\infty\}$.

Note that for all $k \ge 0$ and all $f \in A$, $\delta((f_i)^k f) - \delta((f_i)^k) \ge \delta_i((f_i)^k f) - \delta((f_i)^k) = \delta_i(f)$, so that $\tilde{\delta}_i \ge \delta_i$. To see the opposite inequality, let $f \in A$ and $d \in \mathbb{Z}$ be such that $d \ge \delta_i(f)$. Then $kd_i + d \ge \delta_i((f_i)^k f)$ for all k. Moreover, (3) implies that for sufficiently large k, $kd_i + d > k\delta_j(f_i) + \delta_j(f) = \delta_j((f_i)^k f)$ for all $j \ne i$. It follows that for sufficiently large k, $\delta((f_i)^k f) \le kd_i + d$, and hence $\delta((f_i)^k f) - \delta((f_i)^k) \le d$. Since $\{\delta((f_i)^k f) - \delta((f_i)^k)\}$ is a decreasing sequence due to (5), it follows that $\tilde{\delta}_i(f) \le d$. With $d = \delta_i(f)$ when $\delta_i(f) \in \mathbb{Z}$, and otherwise letting d converge to $\delta_i(f)$ from above, we see that $\tilde{\delta}_i(f) \le \delta_i(f)$. It follows that $\delta_i(f) = \tilde{\delta}_i(f)$, which completes the proof of the lemma and therefore, assertion 3 of theorem 4.1. □

Corollary 4.4. *Let δ be a non-negative subdegree on A and $\delta = \max_{i=1}^N \delta_i$ be its minimal presentation. Then the number of the irreducible components of $X_\infty := X^\delta \setminus X$ is N if none of the δ_i's is the* zero *degree-like function and $N-1$ otherwise.*

Proof. According to proposition 2.2, $X_\infty = V(I) \subseteq \operatorname{Proj} A^\delta$, where I is the ideal in A^δ generated by $(1)_1$. According to lemma 4.1.1, I has a minimal prime decomposition of the form $I = \bigcap_{i=1}^N \mathfrak{p}_i$, with each $\mathfrak{p}_i$ being a prime ideal corresponding to δ_i, $1 \le i \le N$. Consequently $X_\infty = V(I) = \bigcup_{i=1}^N V(\mathfrak{p}_i)$. The corollary now follows from the following

Claim. *For each i, $1 \le i \le N$, the semidegree δ_i is identically zero on A iff $V(\mathfrak{p}_i) = \emptyset$.*

Proof. Let $A^\delta_+ = \bigoplus_{d\ge1}\{(f)_d : f \in A\}$ be the *irrelevant ideal* of A^δ. Fix an i, $1 \le i \le N$. Then

$$\begin{aligned} V(\mathfrak{p}_i) = \emptyset &\iff \mathfrak{p}_i \supseteq A^\delta_+ \\ &\iff \{(f)_d : f \in A,\ d > \delta_i(f)\} \supseteq \{(f)_d : f \in A,\ d \ge 1\} \text{ (according to lemma 4.1.1)} \\ &\iff \delta_i(f) \le 0 \text{ for all } f \in A \\ &\iff \delta = \max\left(\{\delta_j : j \ne i,\ 1 \le j \le N\} \cup \{\delta_0\}\right) \text{ (since } \delta \text{ is non-negative),} \end{aligned}$$

where δ_0 is the zero degree-like function on A, i.e. $\delta_0(f) := 0$ for all $f \in A$. Since δ_0 is a semidegree and the minimal decomposition of δ is unique, it follows that $V(\mathfrak{p}_i) = \emptyset \iff \delta_i = \delta_0$, which completes the proof of the claim. $\square$

Example 4.5. Let $X = (\mathbb{K}^*)^n$ and let $\mathcal{P}$ be a convex integral polytope of dimension n containing the origin in its interior. We saw in example 3.4 that the toric variety $X_{\mathcal{P}}$ is isomorphic to $X^{\delta_{\mathcal{P}}}$ for a subdegree $\delta_{\mathcal{P}}$ on $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Moreover, the minimal presentation of $\delta_{\mathcal{P}}$ is of the form: $\delta_{\mathcal{P}} = \max\{\delta_{\mathcal{Q}} : \mathcal{Q} \text{ is a facet of } \mathcal{P}\}$. Therefore, due to corollary 4.4, we recover the standard fact that the components of $X_{\mathcal{P}} \setminus (\mathbb{K}^*)^n$ are in a one-to-one correspondence with the facets of $\mathcal{P}$ [6, Chapter 3].

4.2 For finitely generated subdegrees, the semidegrees of the minimal presentation do not take $-\infty$ value

In the proof of the next two theorems we make an essential use of the theory of *Rees valuation*. We first describe following [14, chapter XI] the relevant results of Rees (starting with a reminder of the notion of a *Krull domain*).

Definition. A domain B is a *Krull domain* iff

1. $B_{\mathfrak{p}}$ is a discrete valuation ring for all height one prime ideals $\mathfrak{p}$ of B, and
2. every non-zero principal ideal of B is the intersection of a finite number of primary ideals of height one.

Every normal Noetherian domain is a Krull domain [13, Section 41]. In particular, the integral closure of A^δ is a Krull domain provided that A^δ is finitely generated.

For an ideal I of a ring R define $\nu_I : R \to \mathbb{N} \cup \{\infty\}$ and $\bar{\nu}_I : R \to \mathbb{Q}_+ \cup \{\infty\}$ by:

$$\nu_I(x) := \sup\{m : x \in I^m\}, \text{ and}$$
$$\bar{\nu}_I(x) := \lim_{m\to\infty} \frac{\nu_I(x^m)}{m},$$

for all $x \in R$. Recall that the *integral closure* $\bar{J}$ of an ideal J of R is the ideal defined by: $\bar{J} := \{x \in R : x \text{ satisfies an equation of the form: } x^s + j_1 x^{s-1} + \cdots + j_s = 0 \text{ with } j_k \in J^k \text{ for all } k = 1, \ldots, s\}$. The following is due to Rees, see e.g. [14, Propositions 11.1 – 11.6]:

Theorem (Rees)**.** *For any ring R and any ideal I of R, $\bar{\nu}_I$ is well defined. Assume R is a Noetherian domain. Then*

(1) there is a positive integer e such that for all $x \in R$, $\bar{\nu}_I(x) \in \frac{1}{e}\mathbb{N}$, and

(2) if $k \geq 0$ is an integer then $\bar{\nu}_I(x) \geq k$ if and only if $x \in \bar{I^k}$, where $\bar{I^k}$ is the integral closure of I^k in R.

(3) Assume in addition that I is a principal ideal generated by u and $\bar{R}$ is an integral extension of R which is a Krull domain. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height 1 prime ideals of $\bar{R}$ containing u. Then for all $x \in R$, $\bar{\nu}_I(x) = \min\{\frac{\nu_i(x)}{e_i} : i = 1, \ldots, r\}$, where for each $i = 1, \ldots, r$, ν_i is the valuation associated with the discrete valuation ring $\bar{R}_{\mathfrak{p}_i}$ and $e_i := \nu_i(u)$.

The following theorem gives a characterization of the semidegrees δ_i associated to a subdegree δ, provided that A^δ is finitely generated. In particular, it states that each δ_i is integer valued (which is not a priori obvious relying on the limit definition of δ_i's in (4)). We start with the following

Lemma 4.6. *If A^δ is Noetherian, then $\delta(f) \in \mathbb{Z}$ for all $f \in A$, i.e. δ does* not *take the value $-\infty$.*

Proof. Recall that A is a domain and hence A^δ is also a domain by lemma 2.1. Let I be the ideal generated by $(1)_1$ in A^δ. Assume that there exists $f \in A$ such that $\delta(f) = -\infty$. Then $(f)_k \in A^\delta$ for all $k \in \mathbb{Z}$, and therefore $(f)_0 \in \bigcap_{k=1}^\infty I^k$ (since for each $k \geq 0$, $(f)_0 = (f)_{-k} \cdot ((1)_1)^k \in I^k$). In particular, $\bigcap_{k=1}^\infty I^k \neq \emptyset$. On the other hand, since $(1)_0 \notin I$, ideal I is a proper ideal of A^δ and therefore, if A^δ is Noetherian, then $\bigcap_{k=1}^\infty I^k = \emptyset$ [2, Corollary 10.18]. It follows that A^δ is not Noetherian. This completes the proof of the lemma. □

Theorem 4.7. *Let δ be a finitely generated subdegree on A with a minimal presentation $\delta = \max_{1\leq i\leq N} \delta_i$. Let B be any Krull domain which is also an integral extension of A^δ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the height one primes of B containing $(1)_1$. For each j, $1 \leq j \leq r$, define a function $\hat{\delta}_j$ on $A \setminus \{0\}$ by*

$$\hat{\delta}_j(f) = \delta(f) - \frac{\nu_j((f)_{\delta(f)})}{e_j}$$

where ν_j is the discrete valuation of the discrete valuation ring $B_{\mathfrak{p}_j}$ and $e_j := \nu_j((1)_1)$. Then for each i, $1 \leq i \leq N$, $\delta_i \equiv \hat{\delta}_j$ for some j, $1 \leq j \leq r$. In particular, semidegrees δ_i are integer valued for all i.

Proof. Let I be the ideal generated by $(1)_1$ in A^δ and $\bar{\nu}_I$ be the Rees' valuation corresponding to I. Let $f \in A$. According to lemma 4.6, $\delta(f) \in \mathbb{Z}$. Since δ is a subdegree, $\delta(f^k) = k\delta(f)$ for all $k \geq 0$ (corollary 4.2) and hence $((f)_{\delta(f)})^k = (f^k)_{k\delta(f)} \notin I$ for all $k \geq 0$. It follows that $\bar{\nu}_I((f)_{\delta(f)}) = 0$. Now assertion 3 of Rees' theorem implies that $\min_{j=1}^r \frac{\nu_j((f)_{\delta(f)})}{e_j} = 0$, where $e_j := \nu_j((1)_1)$ for every j. Therefore $\delta(f) = \delta(f) - \min_{j=1}^r \frac{\nu_j((f)_{\delta(f)})}{e_j} = \max_{j=1}^r \hat{\delta}_j(f)$ for all $f \in A$. Next we show that for each j, function $e_j\hat{\delta}_j$ is a $\mathbb{Z}$-*valued* semidegree, which suffices to complete the proof of theorem 4.7 due to the uniqueness of the minimal presentation of subdegrees (assertion 3 of theorem 4.1) applied to the minimal presentation $e\delta = \max_i e\delta_i$ and the presentation $e\delta = \max_j e\hat{\delta}_j$ with an appropriate $e \in \mathbb{Z}_+$ (e.g. $e = \prod_j e_j$).

Multiplicativity: Fix j, $1 \leq j \leq r$. Let $f, g \in A \setminus \{0\}$. First we show that $\hat{\delta}_j(fg) = \hat{\delta}_j(f) + \hat{\delta}_j(g)$. Let $\epsilon := \delta(f) + \delta(g) - \delta(fg)$. Then

$$\begin{aligned}\hat{\delta}_j(fg) &= \delta(fg) - \frac{1}{e_j}\nu_j((fg)_{\delta(fg)}) = \delta(f) + \delta(g) - \frac{1}{e_j}(e_j\epsilon + \nu_j((fg)_{\delta(fg)}))\\ &= \delta(f) + \delta(g) - \frac{1}{e_j}(\nu_j((1)_\epsilon) + \nu_j((fg)_{\delta(fg)}))\\ &= \delta(f) + \delta(g) - \frac{1}{e_j}\nu_j((fg)_{\delta(fg)+\epsilon}) = \delta(f) + \delta(g) - \frac{1}{e_j}\nu_j((fg)_{\delta(f)+\delta(g)})\\ &= \delta(f) + \delta(g) - \frac{1}{e_j}(\nu_j((f)_{\delta(f)}) + \nu_j((g)_{\delta(g)})) = \hat{\delta}_j(f) + \hat{\delta}_j(g)\ .\end{aligned}$$

Additivity: Let $d := \delta(f) - \delta(g)$. Note that

$$\begin{aligned} \hat{\delta}_j(f) \geq \hat{\delta}_j(g) &\iff \delta(f) - \frac{\nu_j((f)_{\delta(f)})}{e_j} \geq \delta(g) - \frac{\nu_j((g)_{\delta(g)})}{e_j} \\ &\iff \nu_j((f)_{\delta(f)}) \leq de_j + \nu_j((g)_{\delta(g)}). \end{aligned} \tag{$*$}$$

Moreover, strict inequality in any one of the expressions of $(*)$ implies strict inequality in the other expressions of $(*)$. The remainder of the proof splits into several cases.

Case 1: $\boldsymbol{d > 0}$. In this case $\delta(f+g) = \delta(f)$ and $(f+g)_{\delta(f+g)} = (f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}$. Then $\hat{\delta}_j(f+g) = \delta(f+g) - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)})$.

Subcase 1(a): $\boldsymbol{\hat{\delta}_j(f) > \hat{\delta}_j(g)}$. In this case due to $(*)$ it follows that $\nu_j((f)_{\delta(f)}) < de_j + \nu_j((g)_{\delta(g)}) = \nu_j(((1)_1)^d) + \nu_j((g)_{\delta(g)}) = \nu_j(((1)_1)^d(g)_{\delta(g)})$. Therefore $\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}) = \nu_j((f)_{\delta(f)})$. Hence $\hat{\delta}_j(f+g) = \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)}) = \hat{\delta}_j(f)$.

Subcase 1(b): $\boldsymbol{\hat{\delta}_j(f) = \hat{\delta}_j(g)}$. As in 1(a) it follows by making use of $(*)$ that $\nu_j((f)_{\delta(f)}) = \nu_j(((1)_1)^d(g)_{\delta(g)})$, and therefore $\nu_j((f)_{\delta(f)} + ((1)_1)^d(g)_{\delta(g)}) \geq \nu_j((f)_{\delta(f)})$. Hence $\hat{\delta}_j(f+g) \leq \delta(f) - \frac{1}{e_j}\nu_j((f)_{\delta(f)}) = \hat{\delta}_j(f)$.

Case 2: $\boldsymbol{d = 0}$. Let $e := \delta(f) = \delta(g)$ and $\epsilon := e - \delta(f+g) \geq 0$. It follows that $\hat{\delta}_j(f+g) = \delta(f+g) - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \epsilon - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)}) = \delta(f) - \frac{1}{e_j}\nu_j(((1)_1)^\epsilon(f+g)_{\delta(f+g)}) = e - \frac{1}{e_j}\nu_j((f+g)_{\delta(f+g)+\epsilon}) = e - \frac{1}{e_j}\nu_j((f+g)_e) = e - \frac{1}{e_j}\nu_j((f)_e + (g)_e)$.

Subcase 2(a): $\boldsymbol{\hat{\delta}_j(f) > \hat{\delta}_j(g)}$. In this case due to $(*)$ with $d = 0$ it follows that $\nu_j((f)_e) < \nu_j((g)_e)$. Therefore $\nu_j((f)_e + (g)_e) = \nu_j((f)_e)$. Hence $\hat{\delta}_j(f+g) = e - \frac{1}{e_j}\nu_j((f)_e) = \hat{\delta}_j(f)$.

Subcase 2(b): $\boldsymbol{\hat{\delta}_j(f) = \hat{\delta}_j(g)}$. As in 2(a) $(*)$ implies that $\nu_j((f)_e) = \nu_j((g)_e)$. Consequently $\nu_j((f)_e + (g)_e) \geq \nu_j((f)_e)$. Hence $\hat{\delta}_j(f+g) \leq e - \frac{1}{e_j}\nu_j((f)_e) = \hat{\delta}_j(f)$.

Combining above conclusions it follows that $\hat{\delta}_j$ is indeed a semidegree. As we remarked earlier in the proof, conclusions of theorem 4.7 now follow from the uniqueness of the minimal presentation of a subdegree. ☐

5 Properties of Subdegree

5.1 Regularity at infinity and the normalizing subdegree

Let X be an affine algebraic variety and δ be a finitely generated subdegree on $A := \mathbb{K}[X]$ with minimal presentation $\delta = \max_{1 \leq i \leq N} \delta_i$. Fix i, $1 \leq i \leq N$, such that δ_i is *not* the zero degree-like function. Theorem 4.7 implies that δ_i is integer-valued. Let d_i be the positive generator of the subgroup of $\mathbb{Z}$ generated by $\{\delta_i(f) : f \in A\}$. Recall that $\nu_i(\cdot) := -\frac{1}{d_i}\delta_i(\cdot)$ is a discrete valuation on $\mathbb{K}(X)$ (remark 3.1) and that δ_i corresponds to an irreducible component V_i of $X_\infty := X^\delta \setminus X$ (corollary 4.4). We next show that the local ring $\mathcal{O}_{V_i, X^\delta}$ of X^δ at V_i is precisely the valuation ring of ν_i. In particular, $\mathcal{O}_{V_i, X^\delta}$ is *regular*, i.e. is a discrete valuation ring.

Proposition 5.1. *Let X, δ and A be as above. For each i such that δ_i is non-zero, the local ring $\mathcal{O}_{V_i,X^\delta}$ is regular and hence is a discrete valuation ring. The valuation associated with $\mathcal{O}_{V_i,X^\delta}$ is precisely $\nu_i(\cdot) := -\frac{1}{d_i}\delta_i(\cdot)$.*

Proof. Fix an i, $1 \leq i \leq N$ such that δ_i is not the zero degree-like function. Let $\mathfrak{p}_i$ be the prime ideal of A^δ corresponding to δ_i. Then $\mathcal{O}_{V_i,X^\delta}$ is the degree zero part of the local ring $A^\delta_{\mathfrak{p}_i}$. Let us identify A^δ with $\sum F_d t^d$. Then

$$\begin{aligned} \mathcal{O}_{V_i,X^\delta} &= \{\frac{ft^{kd}}{(gt^d)^k} : gt^d \notin \mathfrak{p}_i,\ d \geq \delta(g),\ k \geq 0\} \\ &= \{\frac{f}{g^k} : (g)_{\delta(g)} \notin \mathfrak{p}_i,\ \delta(f) \leq k\delta(g),\ k \geq 0\}. \end{aligned} \tag{6}$$

Let $R_i \subseteq \mathbb{K}(X)$ be the valuation ring of the discrete valuation ν_i. We have to show that $\mathcal{O}_{V_i,X^\delta} = R_i$. Recall that $R_i := \{g_1/g_2 : g_1, g_2 \in A,\ g_2 \neq 0,\ \nu_i(g_1/g_2) \geq 0\}$. Pick $g_1, g_2 \in A$ such that $g_1/g_2 \in R_i$. Then $\nu_i(g_1/g_2) = \nu_i(g_1) - \nu_i(g_2) \geq 0$ and hence $\delta_i(g_1) \leq \delta_i(g_2)$. Pick $f_i \in A$ such that $\delta_i(f_i) > \delta_j(f_i)$ for all $j \neq i$. It follows due to (4) that there is $k \geq 1$ such that $\delta(g_l f_i^k) = \delta_i(g_l f_i^k)$ for $l = 1, 2$. Then $\delta(g_1 f_i^k) = \delta_i(g_1 f_i^k) = \delta_i(g_1) + \delta_i(f_i^k) \leq \delta_i(g_2) + \delta_i(f_i^k) = \delta_i(g_2 f_i^k) = \delta(g_2 f_i^k)$. Moreover, lemma 4.1.1 implies that $(g_l f_i^k)_{\delta(g_l f_i^k)} \notin \mathfrak{p}_i$ for $l = 1, 2$. Then, according to (6), $g_1 f_i^k/(g_2 f_i^k) = g_1/g_2 \in \mathcal{O}_{V_i,X^\delta}$. Consequently $R_i \subseteq \mathcal{O}_{V_i,X^\delta}$. Therefore $\mathcal{O}_{V_i,X^\delta} = R_i$ due to

Lemma 5.1.1. *Let R be a discrete valuation ring and K be the quotient field of R. If S is a proper subring of K such that $R \subseteq S$, then $R = S$.*

Proof. Let ν be the discrete valuation associated to R and $h \in R$ be a parameter for ν, in particular $\nu(h) = 1$. Assume contrary to the claim that $R \neq S$. Let $f \in S \setminus R$. Then $f = u/h^k$ for some unit u of R and $k > 0$. It follows that $h^{-1} = u^{-1} f h^{k-1} \in S$. Let $g \in K \setminus \{0\}$. Then $\nu(gh^{-\nu(g)}) = 0$ and therefore $gh^{-\nu(g)} \in R \subseteq S$ and also $g = gh^{-\nu(g)} \cdot h^{\nu(g)} \in S$. Therefore $S = K$ contrary to the assumptions, which completes the proof. □

To summarize, we have proved that $\mathcal{O}_{V_i,X^\delta}$ is precisely the valuation ring of ν_i. Since valuations are completely determined by their valuation rings [22, Section VI.8], it follows that ν_i is the valuation corresponding to $\mathcal{O}_{V_i,X^\delta}$, which completes the proof of the proposition. □

Remark 5.2. If V is a codimension one irreducible subvariety a variety Y, then $\mathcal{O}_{V,Y}$ is a regular local ring iff $V \not\subseteq \operatorname{Sing} Y$ iff codimension of $\operatorname{Sing} Y \cap V$ in Y is at least 2 (where $\operatorname{Sing} Y$ is the set of singular points of Y). Therefore, an equivalent formulation of proposition 5.1 is to say that in a completion X^δ of an affine variety X determined by a subdegree δ, the codimension of the 'singular points at infinity' is at least two, or in other words, X^δ is *non-singular in codimension one at infinity.*

Example 5.3. As in example 4.5, let $\delta = \delta_{\mathcal{P}}$ for a convex integral polytope $\mathcal{P}$ of dimension n containing the origin in its interior. Recall that the minimal presentation of $\delta_{\mathcal{P}}$ is $\delta_{\mathcal{P}} = \max\{\delta_{\mathcal{Q}} : \mathcal{Q}$ is a facet of $\mathcal{P}\}$, where for every facet $\mathcal{Q}$ of $\mathcal{P}$,

$$\begin{aligned} \delta_{\mathcal{Q}}(x^\alpha) &:= k\frac{\langle \omega_{\mathcal{Q}}, \alpha\rangle}{c_{\mathcal{Q}}} \text{ for a suitable } k \in \mathbb{N}, \\ \omega_{\mathcal{Q}} &:= \text{the smallest `outward pointing' integral vector normal to } \mathcal{Q}, \text{ and} \\ c_{\mathcal{Q}} &:= \langle \omega_{\mathcal{Q}}, \alpha\rangle \text{ for any } \alpha \text{ in the hyperplane spanned by } \mathcal{Q}. \end{aligned}$$

Since the greatest common divisor of the components of $\omega_{\mathcal{Q}}$ is 1, it follows that the positive greatest common divisor of $\{\delta_{\mathcal{Q}}(f) : f \in A\}$ is $d_{\mathcal{Q}} := \frac{k}{c_{\mathcal{Q}}}$. Therefore, due to proposition 5.1, the order of zero of x^{α} along the component of the hypersurface at infinity corresponding to $\mathcal{Q}$ is $\operatorname{ord}_{\mathcal{Q}}(x^{\alpha}) = -\delta_{\mathcal{Q}}(x^{\alpha})/d_{\mathcal{Q}} = -\langle \omega_{\mathcal{Q}}, \alpha \rangle$ (cf. [6, Section 3.3]).

Let X and δ be as in proposition 5.1. If X is normal, then proposition 5.1 implies that X^{δ} is non-singular in codimension one, i.e. X^{δ} satisfies one of the two criteria of Serre for normality (see, e.g. [13, Theorem 39]). As the following proposition shows, X^{δ} is indeed normal. In fact, X^{δ} is more than just normal - it is *projectively normal*, i.e. A^{δ} is integrally closed.

Proposition 5.4. *If A is an integrally closed domain and δ is a subdegree on A, then A^{δ} is also integrally closed. In particular, if X is a normal affine variety and δ is a complete subdegree on the ring of regular functions of X, then X^{δ} is projectively normal.*

Proof. In (1) we introduced an isomorphism $A^{\delta} \cong \bigoplus_{i \in \mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}]$, with $(1)_1$ being mapped to t. Since A is an integrally closed domain, it follows that $A[t, t^{-1}]$ is also integrally closed [2, Exercise 5.9]. So it suffices to show that A^{δ} is integrally closed in $A[t, t^{-1}]$. Pick $f = \sum_{i=q}^{r} f_i t^i \in A[t, t^{-1}]$ integral over A^{δ}, where $f_i \in A$ for each i. Then f satisfies an equation of the form

$$T^m + G_1 T^{m-1} + \cdots + G_m = 0 \tag{7}$$

for $G_1, \ldots, G_m \in A^{\delta}$. Taking the highest degree terms in t, we see that $f_r t^r$ is integral over f. Replacing f by $f - f_r t^r$ and repeating the procedure, it follows that each $f_i t^i$ is integral over A^{δ}. Therefore it suffices to show that if $f t^k \in A[t, t^{-1}]$ is integral over A^{δ} for some $f \in A$, then $f t^k \in A^{\delta}$. To that end, let $f \in A$ be such that $f t^k$ satisfies an equation of form (7). Comparing the coefficients at t^{km} in that equation, we may assume that $G_i = g_i t^{ik}$ for some $g_i \in A$. Since $g_i t^{ik} \in A^{\delta}$, it follows that $\delta(g_i) \leq ik$.

Assume contrary to the assertion of the proposition that $f t^k \notin A^{\delta}$. Then $d := \delta(f) > k$. Set $T = f t^k$, $G_i = g_i t^{ik}$ and $t = 1$ in (7). It follows that $f^m = -\sum_{i=1}^{m} g_i f^{m-i}$ in A. For each $i \geq 1$, $\delta(g_i f^{m-i}) \leq \delta(g_i) + \delta(f^{m-i}) \leq ik + (m-i)d < id + (m-i)d = md$. Therefore $\delta(f^m) = \delta(-\sum_{i=1}^{m} g_i f^{m-i}) \leq \max_{i=1}^{m} \delta(g_i f^{m-i}) < md$. On the other hand, since δ is a subdegree, $\delta(f^m) = m\delta(f) = md$. The contradiction we arrived at proves $f t^k \in A^{\delta}$, as required. □

Example 5.5. The converse of proposition 5.4 is not true. Indeed, let $A := \mathbb{K}[x]$ and δ be a degree-like function on A defined by

$$\delta(x^k) := \begin{cases} 3k/2 & \text{if } k \text{ is even,} \\ 3(k-1)/2 + 2 & \text{if } k \text{ is odd.} \end{cases}$$

Then $A^{\delta} = \mathbb{K}[(1)_1, (x)_2, (x^2)_3] \cong \mathbb{K}[x, y, z]/\langle x^2 - yz \rangle$, where the isomorphism is induced by the $\mathbb{K}$-algebra homomorphism $\mathbb{K}[x, y, z] \to A^{\delta}$ which sends $x \mapsto (x)_2$, $y \mapsto (x^2)_3$ and $z \mapsto (1)_1$. If $\mathbb{K}$ is not of characteristic 2, then A^{δ} is integrally closed [10, Exercise II.6.4]. On the other hand, $\delta(x^2) = 3 < 4 = 2\delta(x)$ and it follows by making use of corollary 4.2 that δ is not a subdegree.

If X is not normal, then clearly X^δ is not normal. But we will see now that all is not lost: X^δ is *relatively normal at infinity with respect to* X.

Definition 5.6. Let Y be an algebraic variety containing X as a (Zariski) dense open subset and $\psi : Z \to Y$ be a morphism of algebraic varieties. We say that Z is *relatively normal at infinity with respect to* X *via* ψ if for any open subset U of Y, $\Gamma(\psi^{-1}(U), \mathcal{O}_Z)$ is integrally closed in $\Gamma(\psi^{-1}(U \cap X), \mathcal{O}_Z)$. When $Y = Z$ and ϕ is clear from the context, we will simply say that Z is relatively normal at infinity with respect to X.

Note that when X is normal, then Y is relatively normal at infinity with respect to X (via the identity map) iff Y is normal.

Proposition 5.7. *Let δ be a finitely generated non-negative subdegree on the ring A of regular functions on an affine variety X. Then X^δ is relatively normal at infinity with respect to X.*

Proof. Let $D(G) := \{Q \in X^{\mathcal{F}} | G \notin Q\} = \operatorname{Spec}(A^{\mathcal{F}})_{(G)}$ be a basic affine open subset of X^δ, where G is a homogeneous element in $A^\delta$of *positive* degree d and $(A^\delta)_{(G)}$ is the subring of $(A^\delta)_G := A^\delta[\frac{1}{G}]$ consisting of degree zero homogeneous elements. Then $G = (g)_d$ for some $g \in A$ with $d \geq \delta(g)$ and therefore $(A^{\mathcal{F}})_{(G)} = \{\frac{f}{g^k} : f \in A,\ \delta(f) \leq kd\}$. In particular, $\frac{1}{g} \in (A^{\mathcal{F}})_{(G)}$. Consequently, the regular functions on $D(G) \cap X$ are generated as a $\mathbb{K}$-algebra by A and $\frac{1}{g}$, i.e. $D(G) \cap X = \operatorname{Spec} A_g$. We now show that $(A^{\mathcal{F}})_{(G)}$ is integrally closed relative to A_g.

If $d > \delta(g)$, then $G = (g)_d((1)_1)^{d-\delta(g)}$, so that $\operatorname{Spec}(A^{\mathcal{F}})_{(G)} \subseteq \operatorname{Spec}(A^{\mathcal{F}})_{((1)_1)} = \operatorname{Spec} A = X$ (proposition 2.2). It follows that $\operatorname{Spec}(A^{\mathcal{F}})_{(G)} = \operatorname{Spec} A_g$, and therefore $(A^{\mathcal{F}})_{(G)} = A_g$. Then it follws straight from the definition that $(A^{\mathcal{F}})_{(G)}$ is integrally closed relative to A_g. So assume $d = \delta(g)$. Let the semidegrees associated to δ be $\delta_1, \ldots, \delta_N$. W.l.o.g. assume that

$$\begin{aligned} &\delta_i(g) = d \text{ for } 1 \leq i \leq m, \text{ and} \\ &\delta_i(g) < d \text{ for } m < i \leq N. \end{aligned}$$

Claim 5.7.1. $(A^{\mathcal{F}})_{(G)} = A_g \cap \bigcap_{i=1}^m \mathcal{O}_{V_i,X^\delta}$, *where for every i, $1 \leq i \leq m$, V_i is the component of the hypersurface at infinity of X^δ corresponding to δ_i and $\mathcal{O}_{V_i,X^\delta}$ is the local ring of X^δ along V_i.*

Proof. At first note that

$$\begin{aligned} (A^{\mathcal{F}})_{(G)} &= \{\frac{(f)_{kd}}{((g)_d)^k} : \delta(f) \leq kd,\ k \geq 0\} \\ &= \{\frac{f}{g^k} \in A_g : \delta(f) \leq kd,\ k \geq 0\} \\ &= A_g \cap \bigcap_{i=1}^N \{\frac{f}{g^k} : f \in A,\ \delta_i(f) \leq kd,\ k \geq 0\} \\ &= A_g \cap \bigcap_{i=1}^m \{\frac{f}{g^k} : f \in A,\ \delta_i(f) \leq k\delta_i(g),\ k \geq 0\} \\ &\quad \cap \bigcap_{i=m+1}^N \{\frac{f}{g^k} : f \in A,\ \delta_i(f) \leq kd,\ k \geq 0\} \end{aligned}$$

$$= A_g \cap \bigcap_{i=1}^{m} \mathcal{O}_{V_i,X^\delta} \cap \bigcap_{i=m+1}^{N} \{\frac{f}{g^k} : f \in A,\ \delta_i(f) \leq kd,\ k \geq 0\}.$$

It follows that $(A^{\mathcal{F}})_{(G)} \subseteq A_g \cap \bigcap_{i=1}^m \mathcal{O}_{V_i,X^\delta}$. It remains to show inclusion in the opposite direction. Let $f \in A$ and $\frac{f}{g^k} \in \bigcap_{i=1}^m \mathcal{O}_{V_i,X^\delta}$. Then $\delta_i(f) \leq k\delta_i(g) = kd$ for all i, $1 \leq i \leq m$. Recall that $\delta_j(g) < d$ for all j with $m+1 \leq j \leq N$. It follows that if l is a sufficiently large integer, then for all j, $m+1 \leq j \leq N$, $\delta_j(fg^l) = \delta_j(f) + l\delta_j(g) \leq (l+k)d$. Pick an integer l as in the preceding sentence. Then $\delta(fg^l) = \max_{j=1}^N \delta_j(fg^l) \leq (l+k)d$, and therefore $\frac{f}{g^k} = \frac{fg^l}{g^{k+l}} \in (A^{\mathcal{F}})_{(G)}$. This implies that $(A^{\mathcal{F}})_{(G)} \supseteq A_g \cap \bigcap_{i=1}^m \mathcal{O}_{V_i,X^\delta}$ and completes the proof of the claim. ☐

Now, note that for all i, $1 \leq i \leq m$, $\mathcal{O}_{V_i,X^\delta}$ is a discrete valuation ring, therefore in particular, integrally closed. It then follows that $(A^{\mathcal{F}})_{(G)}$ is integrally closed relative to A_g and this completes the proof of the proposition. ☐

Remark 5.8. In the same way that normality implies non-singularity in codimension one, it can be shown that for a completion $\bar{X}$ of X, if $\bar{X}$ is normal at infinity with respect to X, then $\bar{X}$ is also non-singular at infinity in codimension one (with respect to X). Therefore proposition 5.7 strengthens proposition 5.1.

In general, completions corresponding to degree-like functions may have arbitrarily bad singularities at infinity. Now we introduce the notion of *normalization at infinity*, which makes the hypersurface at infinity of a given completion a bit 'less singular', in the same sense that normalization of a singular variety is less singular than itself.

Definition 5.9. Let Y be a variety containing X as a dense open subset. The *normalization of* Y *at infinity with respect to* X is another variety $\tilde{Y}$ containing X as a dense open subset and a finite morphism $\phi : \tilde{Y} \to Y$ such that $\tilde{Y}$ is normal at infinity with respect to X and $\phi|_X$ is the identity map.

Proposition 5.10. *Let X be an arbitrary irreducible variety and Y be a variety containing X as a Zariski dense open subset. Then the normalization $\tilde{Y}$ of Y at infinity (with respect to X) exists and is unique. Moreover, $\tilde{Y}$ has the following universal properties:*

1. *If $\psi : Z \to Y$ is a dominant morphism of algebraic varieties such that Z is normal at infinity with respect to X via ψ, then there exists a unique morphism $\theta : Z \to \tilde{Y}$ such that the following diagram commutes.*

$$\begin{tikzcd} & Z \arrow[dl, "\theta"'] \arrow[dr, "\psi"] & \\ \tilde{Y} \arrow[rr, "\phi"] & & Y \end{tikzcd}$$

2. *If Z is another variety containing X as an open subset and $\psi : Z \to Y$ is a finite morphism such that $\psi|_X$ is the identity map, then there exists a unique morphism $\theta : \tilde{Y} \to Z$ such that the following diagram commutes.*

$$\begin{tikzcd} & Z \arrow[dr, "\psi"] & \\ \tilde{Y} \arrow[ur, "\theta"] \arrow[rr, "\phi"] & & Y \end{tikzcd}$$

Remark 5.11. The proof that follows is completely analogous to the standard proofs of the existence and universal properties of normalization (e.g. the proof in [20]). We nonetheless spell it out for the sake of completion.

Proof. At first we prove the proposition for the case that Y is an affine variety. So let $Y := \operatorname{Spec} B$. Define $\tilde{B}$ to be the integral closure of B in $A := \Gamma(X, \mathcal{O}_X) \subseteq \mathbb{K}(Y)$. Since $\tilde{B}$ is a B-submodule of the integral closure $\bar{B}$ of B in its quotient ring, and since $\bar{B}$ is a finite B-module (see, e.g. [5, Corollary 13.13]), it follows that $\tilde{B}$ is also a finite B-module and therefore is a finitely generated $\mathbb{K}$-algebra. Let $\tilde{Y} := \operatorname{Spec} \tilde{B}$. We claim that $\tilde{Y}$ is a normalization of Y of Y with respect to X. Indeed, pick $f \in B$ such that $Y_f := \operatorname{Spec} B_f \subseteq X$. Then $A_f \subseteq B_f$. On the other hand, $B_f \subseteq (\tilde{B})_f \subseteq A_f$. The two preceding observations together imply that $\tilde{B}_f = B_f = A_f$. Since X has an open cover by subsets of the form $\{\operatorname{Spec} A_f : f \in A\}$, it follows that X is naturally embedded into $\tilde{Y}$ as a Zariski open subset and the finite morphism $\phi : \tilde{Y} \to Y$ induced by the inclusion $B \subseteq \tilde{B}$ is identity on X. Moreover, it follows from the construction of $\tilde{B}$ that $\tilde{Y}$ is normal at infinity with respect to X and therefore $\tilde{Y}$ is a normalization of Y at infinity with respect to X. The uniqueness of $\tilde{Y}$ follows from either of the two universal properties. So it suffices to show the universal properties of the normalization (to prove the proposition for the case that Y is affine).

Let $\psi : Z \to Y$ is a dominant morphism of algebraic varieties such that Z is normal at infinity with respect to X via ψ. Since ψ is dominant, $\psi^* : \mathbb{K}(Y) \to \mathbb{K}(Z)$ is a well-defined injective ring homomorphism. Recall that $\tilde{B}$ is integral over B. Consequently, $\psi^*(\tilde{B})$ is integral over $\psi^*(B)$ and therefore over $\Gamma(Z, \mathcal{O}_Z)$. Since $\psi^*(\tilde{B})$ is also included in $\Gamma(\psi^{-1}(X), \mathcal{O}_Z)$ and since $\Gamma(Z, \mathcal{O}_Z)$ is integrally closed in $\Gamma(\psi^{-1}(X), \mathcal{O}_Z)$, it follows that $\psi^*(\tilde{B}) \subseteq \Gamma(Z, \mathcal{O}_Z)$. Therefore ψ^* induces a morphism $Z \to \tilde{Y}$ which is a lift of ψ and proves the first universal property.

For the second property, assume that Z is another variety containing X as an open subset and $\psi : Z \to Y$ is a finite morphism such that $\psi|_X$ is the identity map. Then $Z = \psi^{-1}(Y)$ is affine (by definition of a finite morphism!) and $\mathbb{K}[Z]$ is integral over $\psi^*(B) = B$ (where the last equality holds because ψ^* is identity on $\mathbb{K}(X) = \mathbb{K}(Y)$). Since X is Zariski open in Z, $\mathbb{K}[Z]$ is also a subset of $\Gamma(X, \mathcal{O}_X)$. Therefore, by the normality at infinity of $\tilde{Y}$, $\mathbb{K}[Z] \subseteq \tilde{B}$. Let $\theta : \tilde{Y} \to Z$ be the morphism induced by the preceding inclusion of rings. Then $\phi = \psi \circ \theta$. It shows that the second universal property also holds and completes the proof of the proposition in the case that Y is affine.

Now assume that Y is an arbitrary variety. Let $\{U_i\}_{i \in \mathcal{I}}$ be an affine open cover of Y. By the affine case, for each $i \in \mathcal{I}$, we may construct the normalization $\phi_i : \tilde{U}_i \to U_i$ at infinity of U_i with respect to $X \cap U_i$. Fix $i, j \in \mathcal{I}$. Both $\phi_i^{-1}(U_i \cap U_j)$ and $\phi_j^{-1}(U_i \cap U_j)$ are normalizations at infinity of $U_i \cap U_j$. Now take an affine open cover $\{V_k\}_{k \in \mathcal{J}}$ of $U_i \cap U_j$. Then there is an isomorphism $\psi_{ijk} : \phi_i^{-1}(V_k) \overset{\cong}{\to} \phi_j^{-1}(V_k) \subseteq \phi_j^{-1}(U_i \cap U_j)$ for each $k \in \mathcal{J}$. Fix $k, k' \in \mathcal{J}$. Since both ψ_{ijk} and $\psi_{ijk'}$ restrict to the identity map on $W_{kk'} := X \cap \phi_i^{-1}(V_k) \cap \phi_i^{-1}(V_{k'})$ and since $W_{kk'}$ is dense in $\phi_i^{-1}(V_k) \cap \phi_i^{-1}(V_{k'})$, it follows that the restrictions of ψ_{ijk} and $\psi_{ijk'}$ on $\phi_i^{-1}(V_k) \cap \phi_i^{-1}(V_{k'})$ are identical. Therefore ψ_{ijk}'s glue together to give an isomorphism $\psi_{ij} : \phi_i^{-1}(U_i \cap U_j) \overset{\cong}{\to} \phi_j^{-1}(U_i \cap U_j)$. Let $\tilde{X}$ be the *scheme* formed by gluing $\{\tilde{U}_i\}_{i \in \mathcal{I}}$ via ψ_{ij}, $i, j \in \mathcal{I}$. Then $\tilde{Y}$ is an reduced irreducible scheme of finite type over $\mathbb{K}$ and $\{\phi_i : i \in \mathcal{I}\}$

glue together to give a *finite* morphism $\phi : \tilde{Y} \to Y$. Since finite morphisms are *separated* [10, Exercise II.5.17], it follows that $\tilde{Y}$ is also a separated scheme, and therefore, a *variety*. Consequently, $\tilde{Y}$ is a normalization of Y at infinity with respect to X. The universal properties (and hence the uniqueness) of the normalization map follows similarly from those for the affine case. ☐

As in proposition 5.10, let X be a Zariski open subset of a variety Y. The proof of proposition 5.10 shows that if Y is affine, then the normalization $\tilde{Y}$ of Y at infinity with respect to X is also affine. If Y is projective, then similarly one can show that $\tilde{Y}$ is also projective. We next show that if X is affine and Y is isomorphic to the completion of X determined by a degree-like function δ on $\mathbb{K}[X]$, then $\tilde{Y}$ is also isomorphic to the completion of X determined by a degree-like function $\tilde{\delta}$. The next theorem gives the construction of one such $\tilde{\delta}$ which is also a *subdegree*.

Theorem 5.12 (Main Existence Theorem). *Let X be an affine variety and δ be a finitely generated degree-like function on $A := \mathbb{K}[X]$. Let I be the ideal generated by $(1)_1$ in A^{δ}. Then*

1. *there is a positive integer e and a subdegree $\tilde{\delta}$ on A such that for all $h \in A$,*
$$\tilde{\delta}(h) := e \lim_{m\to\infty} \frac{\delta(h^m)}{m}.$$
2. *There is a natural inclusion $A^{e\delta} \subseteq A^{\tilde{\delta}}$ of graded rings such that $A^{\tilde{\delta}}$ is integral over $A^{e\delta}$. In particular, $\tilde{\delta}$ is finitely generated. If δ is non-negative (resp. complete), then $\tilde{\delta}$ is also non-negative (resp. complete).*
3. *The variety $X^{\tilde{\delta}}$ is the normalization at infinity of X^{δ} with respect to X.*

Proof. Fix $h \in A$ and $m \in \mathbb{N}$. Then $\delta(h^m) \leq m\delta(h)$. Moreover, since ideal I is generated by $(1)_1$, it follows that $k := m\delta(h) - \delta(h^m)$ is the largest integer such that $(h^m)_{m\delta(h)} \in I^k$. The definition of ν_I implies that $\nu_I(((h)_{\delta(h)})^m) = k = m\delta(h) - \delta(h^m)$. Therefore $\delta(h^m)/m = \delta(h) - \nu_I(((h)_{\delta(h)})^m)/m$. It follows according to assertion 1 of Rees' theorem that $\bar{\delta}(h) := \lim_{m\to\infty} \delta(h^m)/m = \delta(h) - \bar{\nu}_I((h)_{\delta(h)})$ is well defined and there exists a positive integer e (independent of h) such that $\bar{\delta}(h) \in \frac{1}{e}\mathbb{Z}$. Taking $\tilde{\delta} := e\bar{\delta}$ proves the displayed formula of assertion 1. Moreover, note that $\tilde{\delta}(h^m) = m\tilde{\delta}(h)$ for all h and m. Therefore, if we show that $A^{\tilde{\delta}}$ is finitely generated, then it follows via corollary 4.2 that $\tilde{\delta}$ is a subdegree. Hence it suffices to prove assertion 2 to complete the proof of assertion 1.

We now prove assertion 2. Let $\mathcal{F} := \{F_d\}_{d\in\mathbb{Z}}$ be the filtration on A corresponding to δ. As usual, we identify A^{δ} with $\bigoplus_{i\in\mathbb{Z}} F_i t^i$. For $m \in \mathbb{Z}$, let $\bar{F}_{\frac{m}{e}} := \{h \in A : \bar{\delta}(h) \leq \frac{m}{e}\}$ and define $A^{\bar{\delta}} := \bigoplus_{m\in\mathbb{Z}} \bar{F}_{\frac{m}{e}} t^{\frac{m}{e}}$. Since $\bar{\delta} \leq \delta$, it follows that $F_k \subseteq \bar{F}_k$ for each $k \in \mathbb{Z}$. Therefore $A^{\delta} \subseteq A^{\bar{\delta}}$. We will make use of the following

Claim 5.12.1. *$A^{\bar{\delta}}$ is integral over A^{δ}.*

Proof. It suffices to show that for each $h \in A$, $ht^{\bar{\delta}(h)}$ is integral over A^{δ}. Pick $h \in A$. Then $ht^{\bar{\delta}(h)}$ is integral over A^{δ} if and only if $\bar{H} := (ht^{\bar{\delta}(h)})^e$ is integral over A^{δ}. Note that $e\bar{\delta}(h) = \bar{\delta}(h^e)$ by construction of $\bar{\delta}$, and therefore $\bar{H} = h^e t^{\bar{\delta}(h^e)}$. Let $H := h^e t^{\delta(h^e)} \in A^{\delta}$ and $k := \bar{\nu}_I(H)$, where I is the ideal generated by $(1)_1$ in A^{δ}. Since $\nu_I(H^m) = m\delta(H) - \delta(H^m)$

and $\delta(H) = \delta(h^e)$, it follows that $k = \delta(h^e) - \bar{\delta}(h^e) = \delta(h^e) - e\bar{\delta}(h)$. Then k is an integer. Hence according to assertion 2 of Rees' theorem, H is in the integral closure of I^k in A^δ, i.e. H satisfies an equation of the form $H^l + G_1 H^{l-1} + \cdots + G_l = 0$, where $G_i \in I^{ik}$ for each i. Comparing the coefficients at $t^{l\delta(h^e)}$ in the above equation, we may assume w.l.o.g. that each G_i is of the form $g_i t^{i\delta(h^e)}$ for some $g_i \in A$. Since $G_i \in I^{ik}$, it follows that $i\delta(h^e) \geq \delta(g_i) + ik$, implying that $g_i t^{i(\delta(h^e)-k)}$ is an element of A^δ. Moreover, in the ring $A^{\bar{\delta}}$, (via embedding $A^\delta \hookrightarrow A^{\bar{\delta}}$) element $H = h^e t^{\delta(h^e)} = h^e t^{\bar{\delta}(h^e)+k} = h^e t^{\bar{\delta}(h^e)} t^k = t^k \bar{H}$. Substituting these values of H and G_i into the equation of integral dependence for H and then canceling a factor of t^{lk} we conclude that $(\bar{H})^l + \sum_{i=1}^{l} g_i t^{i(\delta(h^e)-k)} (\bar{H})^{l-i} = 0$. But then $\bar{H}$ is integral over A^δ, which completes the proof of the claim. □

Let $\tilde{\mathcal{F}} := \{\tilde{F}_d\}_{d\in\mathbb{Z}}$ be the filtration corresponding to $\tilde{\delta} = e\bar{\delta}$. Observe that $\tilde{F}_d = \{f : e\bar{\delta}(h) \leq d\} = \bar{F}_{\frac{d}{e}}$, and it follows that the homomorphism $\chi : A^{\bar{\delta}} := \bigoplus_{d\in\mathbb{Z}} \bar{F}_{\frac{d}{e}} t^{\frac{d}{e}} \longrightarrow \bigoplus_{d\in\mathbb{Z}} \tilde{F}_d s^d \cong A^{\tilde{\delta}}$ that sends $t \mapsto s^e$ and is the identity map on the coefficients (i.e. on $\bar{F}_{\frac{d}{e}}$ for $d \in \mathbb{Z}$) is an isomorphism of $\mathbb{K}$-algebras. Therefore, it follows due to 5.12.1 that $A^{\tilde{\delta}} = \chi(A^{\bar{\delta}})$ is integral over $\chi(A^\delta)$. On the other hand, since $\tilde{\delta} \leq e\delta$, there is a natural inclusion $A^{e\delta} \subseteq A^{\tilde{\delta}}$ of graded rings and $\chi(A^\delta) \subseteq A^{e\delta}$. Therefore $A^{\tilde{\delta}}$ is integral over $A^{e\delta}$. Since A^δ (and therefore also $A^{e\delta}$) is a finitely generated $\mathbb{K}$-algebra, it follows that $A^{\tilde{\delta}}$ is a finitely generated $\mathbb{K}$-algebra.

If δ is non-negative (resp. complete), then by construction $\tilde{\delta}$ is also non-negative (resp. complete). This completes the proof of assertion 2 and therefore also assertion 1 of the theorem. Moreover, proposition 5.7 implies that $X^{\tilde{\delta}}$ is normal at infinity with respect to X. Since the natural morphism $X^{\tilde{\delta}} \to X^\delta$ induced by the integral inclusion $A^{e\delta} \subseteq A^{\tilde{\delta}}$ is finite, it follows that $X^{\tilde{\delta}}$ is the normalization of X^δ at infinity with respect to X. This completes the proof of the theorem. □

Remark - definition 5.13. Let X, A, δ and $\tilde{\delta}$ be as in theorem 5.12. If in addition X is normal, then according to proposition 5.4 $X^{\tilde{\delta}}$ is also normal. The universal property of the normalization then implies that $X^{\tilde{\delta}}$ is in fact the normalization of X^δ. Motivated by this, we will refer to $\tilde{\delta}$ as a *normalization* of δ.

Example 5.14. Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$ such that $\mathbb{Z}^n = \mathbb{Z}\mathcal{A}$ and the convex hull $\mathcal{P}$ of $\mathcal{A}$ in $\mathbb{R}^n$ contains the origin in its interior. There is a (possibly non-normal) toric variety $X_\mathcal{A}$ corresponding to $\mathcal{A}$ (see e.g. [8, Section 5.1]) - $X_\mathcal{A}$ is the closure of the image of the map $\phi_\mathcal{A} : (\mathbb{K}^*)^n \hookrightarrow \mathbb{P}^{|\mathcal{A}|-1}(\mathbb{K})$ whose components are the monomials x^α with $\alpha \in \mathcal{A}$. Let η be the degree-like function on A corresponding to the completion $(\mathbb{K}^*)^n \hookrightarrow X_\mathcal{A}$ and $\delta_\mathcal{P}$ be a subdegree associated to $\mathcal{P}$ (as in example 3.4). We claim that $\delta_\mathcal{P}$ is a *normalization* of η. Once verified, remark-definition 5.13 would imply that $X_\mathcal{P}$ is the normalization of $X_\mathcal{A}$ (cf. [8, Proposition 2.8(a)]). Indeed, by definition $\delta(\sum a_\alpha x^\alpha) := e \max_{a_\alpha \neq 0} e\delta'(x^\alpha)$, where $\delta'(x^\alpha) := \inf\{r \in \mathbb{Q}_+ : \alpha \in r\mathcal{P}\}$ and e is a suitable positive integer chosen to ensure that δ is integer valued. Moreover, the definition of η implies that

$$\eta(x^\alpha) = \min\{k : \exists\ \alpha_1, \ldots, \alpha_k \in \mathcal{A} \text{ such that } \alpha = \sum_{i=1}^{k} \alpha_i\}, \text{ and}$$

$$\eta(\sum a_\alpha x^\alpha) = \max_{a_\alpha \neq 0} \eta(x^\alpha).$$

Note that if $\eta(x^\alpha) \leq k$ then $\alpha \in k\mathcal{P}$. Hence $\delta_\mathcal{P} \leq e\eta$. Since $\delta_\mathcal{P}$ is a subdegree, it follows that $\delta_\mathcal{P} \leq e\bar{\eta}$, where as in theorem 5.12, $\bar{\eta}(h) := \lim_{m\to\infty} \eta(h^m)/m$ for all $h \in A$. Let $\alpha \in \mathbb{Z}^n$ and $d := \delta_\mathcal{P}(x^\alpha)$. Then $\alpha \in \frac{d}{e}\mathcal{P}$ and hence there is an expression of the form: $\alpha = \frac{d}{e}\sum r_i\alpha_i$ such that $\alpha_i \in \mathcal{A}$ for each i and r_i's are positive rational numbers such that $\sum_i r_i \leq 1$. Let $k \in \mathbb{N}$ be such that kdr_i is an integer for all i. Then $ke\alpha = \sum_i kdr_i\alpha_i$, and the definition of η implies that $\eta(x^{ke\alpha}) \leq \sum_i kdr_i = kd$. Therefore $\bar{\eta}(x^\alpha) \leq \frac{kd}{ke} = \frac{d}{e} = \frac{1}{e}\delta_\mathcal{P}(x^\alpha)$, so that $\delta_\mathcal{P}(x^\alpha) \geq e\bar{\eta}(x^\alpha)$. It follows that $\delta_\mathcal{P} = e\bar{\eta}$ and $\delta_\mathcal{P}$ is the normalization of η, as claimed.

Let δ be a finitely generated subdegree on A and $\delta = \max\{\delta_1, \ldots, \delta_N\}$ be the minimal presentation of δ. For each j, $1 \leq j \leq N$, δ_j can be uniquely extended to $\mathbb{K}(X)$ (since $-\delta_j$ is a valuation on $\mathbb{K}(X)$). It follows that the maximum of δ_j's, $1 \leq j \leq N$, is a subdegree which extends δ to all of $\mathbb{K}(X)$. By an abuse of notation we refer to this extension also by δ. We finish this subsection with a description of the normalization of X^δ in terms of (the extension of) δ.

Proposition 5.15. *Let δ be a finitely generated subdegree on A. If $\tilde{X}$ is the normalization of X, then $\tilde{X}^\delta$ is the normalization of X^δ.*

Proof. Identify A^δ with $\sum_{i\in\mathbb{Z}} F_i t^i \subseteq A[t, t^{-1}]$ as in (1). Let R be the integral closure of A^δ. Then R is also a graded ring and the grading of R is compatible with the grading of A^δ [22, Theorem VII.11]. Moreover, $\tilde{A} := R_{(t)}$ is the integral closure of $A^\delta_{(t)} = A$ and therefore we may identify $\tilde{X}$ with $\operatorname{Spec}\tilde{A}$. On the other hand, since R is a domain and t has degree 1, it follows from remark 2.3 and the constructions from the proof of proposition 2.2 that $R \cong \tilde{A}^\eta$ for a degree-like function η on $\tilde{A}$ with $\eta(f) := \min\{k : f = F/t^k \text{ for some } F \in R_k\}$ for each $f \in \tilde{A}$.

Claim 5.15.1. *The restriction of η to A agrees with δ.*

Proof. Since $\tilde{A}^\eta \supseteq A^\delta$, it follows that for each $f \in A$, $\eta(f) \leq \delta(f)$. Assume there exists $f \in A$ such that $\eta(f) < \delta(f)$. Then $ft^{e-1} \in R$, where $e := \delta(f)$. ft^{e-1} satisfies an equation of the form $(ft^{e-1})^l + \sum_{i=1}^l h_i t^{(e-1)i}(ft^{e-1})^{l-i} = 0$ for $h_1t^{e-1}, \ldots, h_l t^{(e-1)l} \in A^\delta$. Multiplying the equation by t^l, we see that $(ft^e)^l = -\sum_{i=1}^l h_i t^{(e-1)i}(ft^e)^{l-i}(t)^i \in A^\delta$ and hence $(ft^e)^l \in I$. Since I is radical (theorem 4.1), it follows that $ft^e \in I$, which implies in turn that $\delta(f) < e$. This contradiction shows that $\eta(f) \geq \delta(f)$ and completes the proof of the claim. ☐

Let $\bar{\eta}$ be a *normalization* of η as in theorem 5.12, i.e.

$$\bar{\eta}(f) := e \lim_{k\to\infty} \eta(f^k)/k,$$

where e is a suitable positive integer as to ensure $\bar{\eta}$ is integer valued. Let the filtrations on $\tilde{A}$ corresponding to η and $\bar{\eta}$ be respectively $\mathcal{G} := \{G_d : d \in \mathbb{Z}\}$ and $\bar{\mathcal{G}} := \{\bar{G}_d : d \in \mathbb{Z}\}$. Arguments as in the proof of theorem 5.12 show that there are integral inclusions $A^\delta \subseteq \tilde{A}^\eta \subseteq \tilde{A}^{\bar{\eta}}$ induced by $\bigoplus_{d\in\mathbb{Z}} F_d t^d \subseteq \bigoplus_{d\in\mathbb{Z}} G_d t^d \subseteq \bigoplus_{d\in\mathbb{Z}} \bar{G}_d s^d$, where the last inclusion is achieved by sending t to s^e. Since $\tilde{A}^\eta \cong R$ is integrally closed, it follows that $\tilde{X}^{\bar{\eta}} := \operatorname{Proj}\tilde{A}^{\bar{\eta}}$ is the normalization of X^δ.

Recall that the minimal prime ideals containing the ideal $I := \langle t \rangle$ of A^δ are $\mathfrak{p}_1, \ldots, \mathfrak{p}_N$. Since $\sqrt{I\tilde{A}^{\bar{\eta}}} = \langle s \rangle =: J$ and the natural mapping $\operatorname{Proj}\tilde{A}^{\bar{\eta}} \to \operatorname{Proj} A^\delta$ is finite, it follows that for each minimal ideal $\mathfrak{q}$ of J, $\mathfrak{q} \cap A^\delta$ is a minimal prime ideal of I. Therefore we may assume that the minimal prime ideals containing J are $\mathfrak{q}_{i,j}$ for $1 \leq i \leq N$ and $1 \leq j \leq N_i$

where $\mathfrak{q}_{i,j} \cap A^\delta = \mathfrak{p}_i$ for all i, j. Let $\bar{\eta}_{i,j}$ be the semidegree on $\tilde{A}$ associated with $\mathfrak{q}_{i,j}$. Then $\bar{\eta} = \max \bar{\eta}_{i,j}$ is the minimal presentation of $\bar{\eta}$.

Claim 5.15.2. *$\bar{\eta}_{i,j}|_A \equiv e\delta_i$ for all i, j.*

Proof. As in lemma 4.1.1, pick $f_i \in A$ such that $\mathfrak{p}_i = (I : f_i t^{\delta(f_i)})$. Then $f_i t^{\delta(f_i)} \in \cap_{i' \neq i} \mathfrak{p}_{i'} \setminus \mathfrak{p}_i$. According to lemma 4.1.2, for all $f \in A$,

$$\delta_i(f) = \lim_{k \to \infty} \delta((f_i)^k f) - \delta((f_i)^k). \tag{8}$$

Since A^δ is finitely generated, $\delta_i(f)$ is an integer for all i, $1 \leq i \leq N$ (according to theorem 4.7). Then it follows from (8) that there is $k_f \in \mathbb{N}$ such that $\delta_i(f) = \delta((f_i)^{k_f} f) - \delta((f_i)^{k_f})$ and hence $(f_i)^{k_f} f t^{\delta((f_i)^{k_f} f)} \not\in \mathfrak{p}_i$. On the other hand, according to claim 5.15.1 and the definition of $\bar{\eta}$, $\bar{\eta}|_A \equiv e\delta$, so that

$$\begin{gathered} f_i t^{\delta(f_i)} = f_i s^{e\delta(f_i)} = f_i s^{\bar{\eta}(f_i)} \quad \text{and} \\ (f_i)^{k_f} f t^{\delta((f_i)^{k_f} f)} = (f_i)^{k_f} f s^{e\delta((f_i)^{k_f} f)} = (f_i)^{k_f} f s^{\bar{\eta}((f_i)^{k_f} f)}. \end{gathered}$$

Since $\mathfrak{q}_{i,j} \cap A^\delta = \mathfrak{p}_i$, neither of the above elements belongs to $\mathfrak{q}_{i,j}$, and therefore according to lemma 4.1.1, $\bar{\eta}_{i,j}(f_i) = \bar{\eta}(f_i)$ and $\bar{\eta}_{i,j}((f_i)^{k_f} f) = \bar{\eta}((f_i)^{k_f} f)$. Hence $\bar{\eta}_{i,j}(f) = \bar{\eta}_{i,j}((f_i)^{k_f} f) - \bar{\eta}_{i,j}((f_i)^{k_f}) = \bar{\eta}((f_i)^{k_f} f) - \bar{\eta}((f_i)^{k_f}) = e\delta((f_i)^{k_f} f) - e\delta((f_i)^{k_f}) = e\delta_i(f)$. ☐

Since every integer-valued semidegree on $\tilde{A}$ is uniquely determined by its restriction to A, claim 5.15.2 implies that $N_i = 1$ for all i, $1 \leq i \leq N$, and consequently, $\bar{\eta} = e\delta$. Then $\tilde{X}^\delta$ is isomorphic to $\tilde{X}^{\bar{\eta}}$ via the e-uple embedding. Since we have already seen that $\tilde{X}^{\bar{\eta}}$ is the normalization of X^δ, it completes the proof of the proposition. ☐

5.2 Divisor at infinity

In the previous subsection we saw that the associated semidegrees of a subdegree are multiples of the orders of vanishing along the components of the hypersurface at infinity of the corresponding completion. We also know (from example 3.5) that the semidegrees contain more information than the orders of vanishing. This extra information is encoded in the *divisor at infinity.*

Let δ be a non-negative degree-like function on A. For a sufficiently large positive integer d, the *d-uple divisor at infinity* $D^\delta_{d,\infty}$ on X^δ is the divisor of $(1)_d$. More precisely, let $U_0 \cup U_1 \cup \cdots \cup U_m$ be an open cover of X^δ such that $U_0 := X$ and $U_j := X^\delta \setminus V((g_j)_{l_j})$ for some $g_j \in A$ and $l_j \geq 0$ for all j, $1 \leq j \leq m$. Let d be a common multiple of $l_1, \ldots, l_m$. Define

$$\begin{aligned} &h_0 := 1, \text{ and} \\ &h_j := \frac{(1)_d}{((g_j)_{l_j})^{d/l_j}} = \frac{1}{(g_j)^{d/l_j}} \text{ for } 1 \leq j \leq m. \end{aligned}$$

Then the local equation of $D^\delta_{d,\infty}$ on U_j is h_j, $0 \leq j \leq m$. Since $V((1)_1) = X_\infty$, it follows that the support of $D^\delta_{d,\infty}$ is X_∞, which justifies index ∞ as a subscript of $D^\delta_{d,\infty}$.

Lemma 5.16. *Let δ be a non-negative finitely generated degree-like function on A and d be a sufficiently large positive integer.*

1. *The sheaf* $\mathcal{O}_{X^\delta}(D^\delta_{d,\infty})$ *of* $D^\delta_{d,\infty}$ *is ample and its global sections are precisely* $\{f \in A : \delta(f) \leq d\}$.
2. *Assume in addition that* δ *is a subdegree and* $\delta_1, \ldots, \delta_N$ *are the non-zero semidegrees associated to* δ. *Then the Weil divisor associated to* $D^\delta_{d,\infty}$ *is*

$$[D^\delta_{d,\infty}] = \sum_{j=1}^{N} \frac{d}{d_j}[X^\delta_{\infty,j}]$$

where for every j, *integer* d_j *is the positive generator of the subgroup of* $\mathbb{Z}$ *generated by* $\{\delta_j(f) : f \in A\}$ *and* $X^\delta_{\infty,j}$ *is the irreducible component of* X_∞ *associated to* δ_j.

Proof. Since $\mathcal{O}_{X^\delta}(D^\delta_{d,\infty})$ is precisely the *twisting sheaf* $\mathcal{O}_{X^\delta}(d)$, assertion 1 follows from standard results in algebraic geometry (see, e.g. [10, Exercise II.5.14]). So we have to prove only assertion 2. Fix an integer j, $1 \leq j \leq N$. Local ring $\mathcal{O}_{X^\delta_{\infty,j}, X^\delta}$ is a discrete valuation ring and its associated valuation is $\nu_j(\cdot) := -\frac{\delta_j(\cdot)}{d_j}$ (proposition 5.1). Pick k, $1 \leq k \leq N$, such that $X^\delta_{\infty,j} \cap U_k \neq \emptyset$. Recall that $U_k := X^\delta \setminus V((g_k)_{l_k})$ and a local equation for $D^\delta_{d,\infty}$ on U_k is $1/(g_k)^{d/l_k}$. Let $\mathfrak{p}_j$ be the ideal of A^δ corresponding to $X^\delta_{\infty,j}$. Since $X^\delta_{\infty,j} \cap U_k \neq \emptyset$, it follows that $(g_k)_{l_k} \notin \mathfrak{p}_j$ and therefore $\delta_j(g_k) = l_k$ according to assertion 1 of lemma 4.1.1. Hence the coefficient of $[X^\delta_{\infty,j}]$ in the expression for $[D^\delta_{j,\infty}]$ is $\nu_j(1/g_k^{d/l_k}) = -\frac{d}{l_k}\nu_j(g_k) = \frac{d}{l_k} \cdot \frac{l_k}{d_j} = \frac{d}{d_j}$. Therefore $[D^\delta_{d,\infty}] = \sum_{j=1}^{N} \frac{d}{d_j}[X^\delta_{\infty,j}]$, which proves assertion 2 and therefore completes the proof of the lemma. $\square$

Example 5.17. We continue with giving illustrations from toric geometry in the notation of example 3.4. Recall (from example 5.3) that for each facet $\mathcal{Q}$ of $\mathcal{P}$, the positive greatest common divisor of $\{\delta_\mathcal{Q}(f) : f \in A\}$ is $d_\mathcal{Q} := \frac{k}{c_\mathcal{Q}}$. Therefore, $[D^{\delta_\mathcal{P}}_{d,\infty}] = \frac{d}{k}\sum_\mathcal{Q} c_\mathcal{Q}[X^{\delta_\mathcal{P}}_{\infty,\mathcal{Q}}]$, where $X^{\delta_\mathcal{P}}_{\infty,\mathcal{Q}}$ is the component of the hypersurface at infinity of $X^{\delta_\mathcal{P}}$ corresponding to $\delta_\mathcal{Q}$. In particular, $D^{\delta_\mathcal{P}}_{d,\infty} = \frac{d}{k}D_\mathcal{P}$, where $D_\mathcal{P}$ is the divisor corresponding to $\mathcal{P}$ [6, Section 3.4].

The next result gives a formula for the *infinite part* of the pull-back of the divisor at infinity under a dominant morphism between completions of affine varieties corresponding to subdegrees.

Proposition 5.18. *Let* δ *be a finitely generated non-negative subdegree and* $\delta_1, \ldots, \delta_N$ *be the non-zero semidegrees associated to* δ. *If* Y *is an affine variety,* η *is a finitely generated non-negative subdegree on* $B := \mathbb{K}[Y]$ *and* $\phi : X^\delta \to Y^\eta$ *is a dominant morphism, then*

$$\phi^*(D^\eta_{d,\infty}) = d\sum_{j=1}^{N} \frac{l^\phi_\infty(\eta, \delta_j)}{d_j}[X^\delta_{\infty,j}] + W, \tag{9}$$

where W *is a Weil divisor on* X^δ *such that* $\mathrm{Supp}(W) = \overline{\mathrm{Supp}(W) \cap X}$, *and for each* j, $1 \leq j \leq N$,

$$l^\phi_\infty(\eta, \delta_j) := \max\left\{\frac{\delta_j(\phi^*(f))}{\eta(f)} : f \in B,\ \eta(f) > 0\right\}, \text{ and}$$
$$d_j := \gcd\{\delta_j(f) : f \in A\},$$

Remark. Identity (9) in particular implies that $l^{\phi}_{\infty}(\eta, \delta_j)$ exists.

Proof. Let the non-zero semidegrees associated to η be $\eta_1, \dots, \eta_M$. For each j, $1 \leq j \leq M$, let $e_j := \gcd\{\eta_j(f) : f \in B\}$, $Y^{\eta}_{\infty,j}$ be the component of the hypersurface at infinity corresponding to η_j, and μ_j is the order of vanishing along $Y^{\eta}_{\infty,j}$. Pick $f \in B$ such that $\eta(f) > 0$. Since f is regular on Y, it follows that

$$
\begin{aligned}
&\operatorname{div}_{Y^{\eta}}(f) - \sum_{j=1}^{M} \mu_j(f)[Y^{\eta}_{\infty,j}] \geq 0 \text{ (where } \operatorname{div}_{Y^{\eta}}(f) \text{ is the principal divisor of } f \text{ on } Y^{\eta}) \\
\Rightarrow\, &\operatorname{div}_{Y^{\eta}}(f) + \frac{1}{d}\sum_{j=1}^{M} \eta_j(f)\frac{d}{e_j}[Y^{\eta}_{\infty,j}] \geq 0 \text{ (according to proposition 5.1)} \\
\Rightarrow\, &\operatorname{div}_{Y^{\eta}}(f) + \frac{\eta(f)}{d}\sum_{j=1}^{M} \frac{d}{e_j}[Y^{\eta}_{\infty,j}] \geq 0 \text{ (since } \eta(f) \geq \eta_j(f) \text{ for all } j,\ 1 \leq j \leq M) \\
\Rightarrow\, &\operatorname{div}_{Y^{\eta}}(f) + \frac{\eta(f)}{d} D^{\eta}_{d,\infty} \geq 0 \text{ (according to lemma 5.16).}
\end{aligned}
$$

Pulling back the divisors that appear in the preceding inequality to X^{δ} (which is possible, since ϕ is dominant and both are Cartier divisors), we see that $\operatorname{div}_{X^{\delta}}(\phi^*(f)) + \frac{\eta(f)}{d}\phi^*(D^{\eta}_{d,\infty}) \geq 0$. Fix an integer j, $1 \leq j \leq N$. Let ν_j be the order of vanishing along $X^{\delta}_{\infty,j}$ and c_j be the coefficient of $[X^{\delta}_{\infty,j}]$ in the expression of $\phi^*(D^{\eta}_{d,\infty})$. It follows that $\nu_j(\phi^*(f)) + \frac{\eta(f)}{d}c_j \geq 0$, or equivalently, $c_j \geq -\frac{d}{\eta(f)}\nu_j(\phi^*(f)) = \frac{d\delta_j(\phi^*(f))}{d_j\eta(f)}$. Moreover, $c_j \geq 0$, since $D^{\eta}_{d,\infty}$ is effective. Consequently, $c_j \geq l^{\phi}_{\infty}(\eta, \delta_j)$.

To complete the proof of the proposition, it suffices to show that there exists $f \in B$ such that $\eta(f) > 0$ and $c_j = \frac{d\delta_j(\phi^*(f))}{d_j\eta(f)}$. We divide the proof in two cases:

Case 1: $\boldsymbol{\phi(X^{\delta}_{\infty,j}) \subseteq Y}$. In this case $c_j = 0$, since $\operatorname{Supp}(D^{\eta}_{d,\infty}) = Y^{\eta}_{\infty}$. On the other hand, for all $f \in B$, $\phi^*(f)$ is regular on $X^{\delta}_{\infty,j}$, so that $\nu_j(\phi^*(f)) \geq 0$, and therefore, $\delta_j(\phi^*(f)) \leq 0$. Now, pick any $f \in B$ such that $\eta(f) > 0$. Since $\delta_j(\phi^*(f)) \leq 0$ and $\delta_j(\alpha) = 0$ for all $\alpha \in \mathbb{K}$, it follows that there exists $\alpha \in \mathbb{K}$ such that $\delta_j(\phi^*(f) + \alpha) = 0$. Consequently, $l^{\phi}_{\infty}(\eta, \delta_j) = \frac{\delta_j(\phi^*(f+\alpha))}{\eta(f+\alpha)} = 0 = c_j$, as required.

Case 2: $\boldsymbol{\phi(X^{\delta}_{\infty,j}) \not\subseteq Y}$. Pick $x \in X^{\delta}_{\infty,j}$ such that $\phi(x) \in Y^{\eta}_{\infty}$. Since $D^{\eta}_{d,\infty}$ is ample, it follows due to assertion 1 of lemma 5.16 that there exists k such that $\mathcal{O}_{Y^{\eta}}(D^{\eta}_{kd,\infty})$ is globally generated by $\{f \in B : \eta(f) \leq kd\}$. In particular, there exists $f \in B$ such that $\eta(f) \leq kd$ and the local equation of fh is invertible near $\phi(x)$, where h is a local equation for $D^{\eta}_{kd,\infty}$. Moreover, since $\phi(x) \in Y^{\eta}_{\infty}$, we may in addition assume that $\eta(f) = kd$. In particular, $\eta(f) > 0$. Now, $\phi^*(fh)$ is invertible near x, which implies that $\nu_j(\phi^*(fh)) = 0$. Note that $\nu_j(\phi^*(fh)) = \nu_j(\phi^*(f)) + \nu_j(\phi^*(h)) = -\frac{\delta_j(\phi^*(f))}{d_j} + kc_j$. Taken together, the preceding two sentences imply that $c_j = \frac{\delta_j(\phi^*(f))}{kd_j} = \frac{d\delta_j(\phi^*(f))}{d_j\eta(f)}$, as required. $\square$

Corollary 5.19. *Let* η*,* δ *and* δ_j*,* $X^{\delta}_{\infty,j}$*,* $1 \leq j \leq N$*, be as in proposition 5.18. Assume in addition that codimension of* $X \setminus \phi^{-1}(Y)$ *in* X *is bigger than or equal to* 2*. Then* $\phi^*(D^{\eta}_{d,\infty}) = d\sum_{j=1}^{N} \frac{l^{\phi}_{\infty}(\eta,\delta_j)}{d_j}[X^{\delta}_{\infty,j}]$*.* □

Remark - definition 5.20. Given a Krull local ring $\mathfrak{o}$, a valuation ω and a height one prime ideal $\mathfrak{p}$ of $\mathfrak{o}$, the *linking number* of ω and the valuation $\nu_{\mathfrak{p}}$ corresponding to on $\mathfrak{p}$ is

$$l(\omega, \nu_{\mathfrak{p}}) := \inf_{f \in \mathfrak{p},\ f \neq 0} \frac{\omega(f)}{\nu_{\mathfrak{p}}(f)}.$$

The linking number was introduced in [19] in connection with defining the pull-back of Weil divisors under a birational regular mapping (more precisely, if $\phi : Z \to Y$ is a birational map and ω (resp. $\nu_{\mathfrak{p}}$) is the order of vanishing along a codimension one subvariety W of Z (resp. V of Y), then $l(\omega, \nu_{\mathfrak{p}})$ is a 'candidate' for the coefficient of $[W]$ in $\phi^*([V])$). In [11] the linking numbers of general pseudo-valuations ω and ν on a ring A was considered, provided ν is *non-negative* on A. In the notation of proposition 5.18, if ϕ is also birational, then

$$l^{\phi}_{\infty}(\eta, \delta_j) = \frac{1}{l(-\delta_j, -\eta)},$$

and moreover, $-\eta$ is *non-positive* on A. We call $l^{\phi}_{\infty}(\eta, \delta_j)$ the *linking number at infinity (relative to* ϕ*)* of η and δ_j. In the case that ϕ is birational (so that ϕ^* is identity on the function field), we simply write $l_{\infty}(\eta, \delta_j)$ for $l^{\phi}_{\infty}(\eta, \delta_j)$.

6 The Case of Dimension 2

6.1 Orders of vanishing along components of the hypersurface at infinity determine the completion

The main results of section 6 are a generalization of Gordan's lemma to arbitrary (finitely generated) positive semidegrees in dimension 2 (theorem 6.9) and invertibility (also in dimension 2) of the matrix of linking numbers of complete subdegrees (proposition 6.12). These are proved in later subsections. In this subsection we study a property of two dimensional varieties which is fundamental to our proofs of those results. As an immediate application we show that, as opposed to higher dimensions, in the case of surfaces the orders of vanishing along components of the hypersurface at infinity determines the completion corresponding to a subdegree (cf. example 3.5). Our first lemma states the most basic form of the said property.

Lemma 6.1. *Let* X *be an affine surface and* $\bar{X}^{(1)}, \ldots, \bar{X}^{(k)}$ *be complete irreducible varieties containing* X*. Let* $\bar{X}$ *be the closure in* $\bar{X}^{(1)} \times \cdots \times \bar{X}^{(k)}$ *of the image of the diagonal embedding of* X*. Then for every irreducible component* C *of* $Z := \bar{X} \setminus X$*, there exists* j*,* $1 \leq j \leq k$*, such that* $\pi_j|_Z$ *is a finite-to-one map of* C *onto an irreducible component of* $Z_j := \bar{X}^{(j)}$*, where* π_j *is the natural projection map* $\bar{X}^{(1)} \times \cdots \times \bar{X}^{(k)} \to \bar{X}^{(j)}$*.*

Proof. Let C be an irreducible component of Z. Since X is affine, it follows that each of $Z, Z_1, \ldots, Z_k$ has pure dimension one (see e.g. [9, Proposition 1]); in particular $\dim C = 1$. Then there exists j, $1 \leq j \leq k$, such that $\dim(\pi_j(C))$ is also 1. Since $\bar{X}^{(1)} \times \cdots \times \bar{X}^{(k)}$ is complete, it follows that $\pi_j(C)$ is a closed irreducible subvariety of Z_j. The assertions of the lemma is now a consequence of the fact that $\dim C = \dim \pi_j(C) = \dim Z_j$. □

We will also frequently use the following property of codimension one subvarieties whose local rings are regular.

Lemma 6.2. *Let W be a codimension one irreducible subvariety of an irreducible variety Z such that $\mathcal{O}_{W,Z}$ is a regular local ring. Then for each surjective birational morphism $\phi : Y \to Z$ is of irreducible algebraic varieties, there exists a unique irreducible subvarity V of Y such that $\phi(V) = W$. Moreover, $\mathcal{O}_{V,Y} = \phi^*(\mathcal{O}_{Z,W})$, so that $\mathcal{O}_{V,Y}$ is also regular. In the case that ϕ is the normalization map, $\phi|_V : V \to W$ is also birational.*

Proof. Let $\phi : Y \to Z$ be a surjective birational morphism of irreducible algebraic varieties. Then there is a codimension one subvariety V of Y such that $\phi(V) = W$. It follows that $\phi^*(\mathcal{O}_{W,Z}) \subseteq \mathcal{O}_{V,Y}$. Since $\phi^* : \mathbb{K}(Z) \to \mathbb{K}(Y)$ is an isomorphism, lemma 5.1.1 implies that $\mathcal{O}_{V,Y} = \phi^*(\mathcal{O}_{W,Z})$. Since V is uniquely determined by $\mathcal{O}_{V,Y}$, it follows that V is unique. This proves the first two assertions of the lemma.

Let $\mathrm{Sing}(Z)$ be the set of singular points of Z. The regularity of $\mathcal{O}_{W,Z}$ implies that $W \not\subseteq \mathrm{Sing}(Z)$. Then the last assertion follows from the observation that, if $\phi : Y \to Z$ is the normalization map, then ϕ restricts to an isomorphism between $Y \setminus \phi^{-1}(\mathrm{Sing}(Z))$ and $Z \setminus \mathrm{Sing}(Z)$. ☐

Corollary 6.3. *Let δ and δ' be finitely generated subdegrees on A and $\psi : X^{\delta'} \to X^{\delta}$ be a surjective morphism which is identity on X. Then for every associated semidegree μ of δ, there is a positive rational number q such that $q\mu$ is an associated semidegree of δ'.*

Proof. Let μ be a semidegree associated to δ, and V be the component of the hypersurface at infinity of X^{δ} corresponding to μ. Since $\mathcal{O}_{V,X^{\delta}}$ is a regular local ring (according to proposition 5.1), lemma 6.2 implies that there exists a unique component V' of $X^{\delta'}_{\infty}$ such that $\psi(V') = V$ and $\mathcal{O}_{V,X^{\delta}} = \mathcal{O}_{V',X^{\delta'}}$. Let μ' be the semidegree associated to δ' which corresponds to V'. Then it follows due to proposition 5.1 that $\mu = q\mu'$ for some rational $q > 0$. ☐

We mainly use lemma 6.1 in the form of the following

Corollary 6.4. *Let X be an affine surface, $\eta_1, \ldots, \eta_k$ be complete subdegrees on $A := \mathbb{K}[X]$ and $\bar{X}$ be the closure in $X^{\eta_1} \times \cdots \times X^{\eta_k}$ of the image of the diagonal embedding of X. Let $\delta_1, \ldots, \delta_m$ be a minimal collection of semidegrees such that for each j, $1 \leq j \leq k$, there exist $r_{j1}, \ldots, r_{jm} \geq 0$ such that $\eta_j = \max\{r_{j1}\delta_1, \ldots, r_{jm}\delta_m\}$. If η is a complete subdegree on A such that X^{η} is isomorphic to the normalization at infinity of $\bar{X}$ (with respect to X), then the minimal presentation of η is of the form:*

$$\eta = \max\{r_1\delta_1, \ldots, r_m\delta_m\} \tag{10}$$

for some $r_1, \ldots, r_m > 0$.

Proof. Recall (from the definition of normalization at infinity) that there is a finite-to-one map $\phi : X^{\eta} \to \bar{X}$ which is identity on X. Let χ be an associated semidegree of η and C be the corresponding component of the curve at infinity of X^{η}. Then $\phi(C)$ is an irreducible component of $\bar{X} \setminus X$ and therefore lemma 6.1 implies that there exists j, $1 \leq j \leq k$, such that $\pi_j(\phi(C))$ is an irreducible component of $Z_j := X^{\eta_j} \setminus X$. Moreover, $\mathcal{O}_{\pi_j(\phi(C)),X^{\eta_j}}$ is regular (according to proposition 5.1) and $(\pi_j \circ \phi)^*$ is the identity map on $\mathbb{K}(X)$. Therefore lemma 6.2 implies that $\mathcal{O}_{\pi_j(\phi(C)),X^{\eta_j}} = \mathcal{O}_{C,X^{\eta}}$. Proposition 5.1 then implies that χ is identical to a rational multiple of the semidegree associated to η_j corresponding to $\pi_j(\phi(C))$. It follows from the assumption on $\delta_1, \ldots, \delta_m$ that $\chi = r_l\delta_l$ for some l, $1 \leq l \leq m$ and $r_l > 0$. Therefore, there exists a presentation (which is a priori *not necessarily* minimal) of η of the form (10).

On the other hand, for each j, $1 \leq j \leq k$, applying corollary 6.3 to morphism $\pi_j \circ \phi : X^\eta \to X^{\eta_j}$ we see that a multiple of each of the assoicated semidegrees of η_j appears in the minimal presentation of η. Therefore (10) is indeed the minimal presentation of η. □

As an application of the preceding considerations we prove the following result which was promised in example 3.5.

Proposition 6.5. *The projective completion of an affine surface corresponding to a complete subdegree is uniquely determined by the orders of vanishing along the components of the curve at infinity.*

Proof. Let X be an affine surface and δ and δ' be two complete subdegrees on $A := \mathbb{K}[X]$ such that the curves at infinity on X^δ and $X^{\delta'}$ have the same number k of irreducible components $C_1, \ldots, C_k$ (respectively $C'_1, \ldots, C'_k$) and for each j, $1 \leq j \leq k$, and for all $f \in A$, the orders of vanishing of f along C_i and C'_i are equal. or equivalently, $\mathcal{O}_{C_j, X^\delta} = \mathcal{O}_{C'_j, X^{\delta'}}$. We have to show that $X^\delta \cong X^{\delta'}$.

Let δ_j (resp. δ'_j) be the associated semidegree of δ (resp. δ') corresponding to C_j (resp. C'_j), $1 \leq j \leq k$. Proposition 5.1 implies that for each j, $1 \leq j \leq k$, $\delta'_j = s_j\delta_j$ for some positive rational number s_j. Let $\bar{X}$ be the closure in $X^\delta \times X^{\delta'}$ of the image of the diagonal embedding of X and η be a normalization of the degree-like function corresponding to the completion $X \hookrightarrow \bar{X}$. Denote the projection map $\bar{X} \to X^\delta$ by π and the *normalization-at-infinity map* $X^\eta \to \bar{X}$ by ϕ. Corollary 6.4 implies that the minimal presentation of η is of the form $\eta = \max\{r_j\delta_j : 1 \leq j \leq k\}$, for some $r_1, \ldots, r_k > 0$. In particular, the number of components of the hypersurface at infinity on X^η is k. It follows that $\pi \circ \phi : X^\eta \to X^\delta$ is a finite map, and therefore, is the normalization at infinity of X^δ with respect to X. But X^δ is itself normal at infinity with respect to X, so that the uniqueness of the normalization at infinity implies that $X^\eta \cong X^\delta$. It follows similarly that $X^\eta \cong X^{\delta'}$. Therefore $X^\delta \cong X^{\delta'}$, as required. □

Remark 6.6. In fact proposition 6.5 remains true for more general subdegrees. More precisely, if δ and δ' are non-negative finitely generated semidegrees on the coordinate ring of an affine surface X such that the orders of vanishing along the components of the curves at infinity on X^δ and $X^{\delta'}$ are identical, then $X^\delta \cong X^{\delta'}$.

6.2 Maxima of finitely generated complete semidegrees are also finitely generated

In this subsection we prove that in dimension 2, the subdegrees determined by finitely generated complete semidegrees are also finitely generated. In the proof we use some standard notions and facts from the theory of polytopes and toric varieties which we now recall. Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^n$. For each nonempty face $\mathcal{F}$ of P, its *inner normal cone* $\mathcal{N}(\mathcal{F}; \mathcal{P})$ is the set of vectors defining linear functions which are minimized over $\mathcal{P}$ on $\mathcal{F}$. If $\mathcal{P}$ has full dimension, the dimension of $\mathcal{N}(\mathcal{F};\mathcal{P})$ is $\dim\mathcal{P}-\dim\mathcal{F}$. The set of all normal cones of a $\mathcal{P}$ is a polyhedral complex of relatively open cones and it is known as the *normal fan* $\mathcal{N}_{\mathcal{P}}$ of $\mathcal{P}$.

Facts. 1. Let $\mathcal{P}_1,\dots,\mathcal{P}_k$ be convex polytopes in $\mathbb{R}^n$ and let $\mathcal{P}:=\mathcal{P}_1+\cdots+\mathcal{P}_k$. The normal fan $\mathcal{N}_{\mathcal{P}}$ of $\mathcal{P}$ is the common refinement of normal fans of the summands, i.e. the set of nonempty intersections of normal cones of the faces of the $\mathcal{P}_j$'s (see e.g. [23, subsection 7.2]).

2. Let $\mathcal{P}$ and $\mathcal{Q}$ be full dimensional lattice polytopes in $\mathbb{R}^n$, i.e. the vertices of $\mathcal{P}$ and $\mathcal{Q}$ lie on a subgroup $\mathcal{L}$ of $\mathbb{R}^n$ such that $\mathcal{L}\cong\mathbb{Z}^n$. Identification of $\mathcal{L}$ with $\mathbb{Z}^n$ induces an identification of $(\mathbb{K}^*)^n$ with the tori in $\mathcal{P}$ and $\mathcal{Q}$. If $\mathcal{N}_{\mathcal{P}}$ is a refinement of $\mathcal{N}_{\mathcal{Q}}$, then there is a morphism $\phi: X_{\mathcal{P}}\to X_{\mathcal{Q}}$ of toric varieties which is identity on $(\mathbb{K}^*)^n$ (see e.g. [6, subsection 1.4]).

3. Let $\mathcal{P}_1,\dots,\mathcal{P}_k$ and $\mathcal{P}$ be as in fact 1. Moreover, assume that these polytopes are *lattice polytopes*. As in fact 2, identify the tori of these toric varieties with $(\mathbb{K}^*)^n$. Then $X_{\mathcal{P}}$ is isomorphic to the closure of the image of $(\mathbb{K}^*)^n$ via the *diagonal* embedding $(\mathbb{K}^*)^n\hookrightarrow X_{\mathcal{P}_1}\times\cdots\times X_{\mathcal{P}_k}$. (This can be seen by determining which monomials appear in the diagonal map. Indeed, there exists m (in fact, all $m\geq n-1$ will do) such that for all j, $1\leq j\leq k$, $X_{\mathcal{P}_j}$ is isomorphic to the closure of the map from $(\mathbb{K}^*)^n$ to a projective space whose coordinates are monomials with exponents belonging to $m\mathcal{P}_j$. Composing the above maps with the *Segre embedding* of $X_{\mathcal{P}_1}\times\cdots\times X_{\mathcal{P}_k}$, we see that the monomials corresponding to the diagonal embedding of $(\mathbb{K}^*)^n$ into $X_{\mathcal{P}_1}\times\cdots\times X_{\mathcal{P}_k}$ are precisely the monomials with exponents belonging to $m\mathcal{P}_1+\cdots+m\mathcal{P}_k=m\mathcal{P}$. Since $X_{\mathcal{P}}$ is by definition the closure of the image of the map with the same coordinates, fact 3 follows.)

For all full dimensional lattice polytopes $\mathcal{P}$ of a fixed $\mathbb{R}^n$, from now on we fix an isomorphism of the lattice with $\mathbb{Z}^n$ and identify (as in fact 2) the tori of the toric varieties $X_{\mathcal{P}}$ with $(\mathbb{K}^*)^n$.

Lemma 6.7. *Let $\Delta_1,\dots,\Delta_k$ be simplices in $\mathbb{R}^n$ such that the vertices of each Δ_i, $1\leq i\leq k$, are the origin and integral points on the positive coordinate axes. Let $\Delta:=\Delta_1+\cdots+\Delta_k$. For every k-tuple $r:=(r_1,\dots,r_k)$ of positive integers, define $\Delta^{(r)}:=\bigcap_{i=1}^k r_i\Delta_i$. Then the normal fan $\mathcal{N}_\Delta$ of Δ is a refinement of the normal fan $\mathcal{N}_{\Delta^{(r)}}$ of $\Delta^{(r)}$ and there is a morphism $X_\Delta\to X_{\Delta^{(r)}}$ which is identity on $(\mathbb{K}^*)^n$.*

Proof. We prove the lemma by induction on $l(r):=$ the number of facets of $\Delta^{(r)}$ which are different from the coordinate hyperplanes. If $l(r)=1$, then $\Delta^{(r)}=r_i\Delta_i$ for some i, $1\leq i\leq k$ and the lemma follows from fact 1. So assume that the lemma holds for all $r'\in(\mathbb{Z}_+)^k$ with $l(r')<l(r)$. Let σ be an arbitrary full dimensional cone in $\mathcal{N}_{\Delta^{(r)}}$.

Claim. *For every cone τ of $\mathcal{N}_\Delta$, if $\tau\cap\sigma\neq\emptyset$, then $\tau\subseteq\sigma$.*

Proof. W.l.o.g. assume $\Delta^{(r)}=\bigcap_{i=1}^l r_i\Delta_i$ where $l:=l(r)$. For each i, $1\leq i\leq r$, let H_i be the hyperplane containing the (only) facet of Δ_i which is not on a coordinate hyperplane, and let η_i be the edge of $\mathcal{N}_{\Delta^{(r)}}$ which is normal to H_i. We divide the proof of the claim in the following two cases:

Case 1: there exists i, $1\leq i\leq l$, such that $\eta_i\not\in\sigma$. W.l.o.g. assume $i=1$. Let v be the vertex of $\Delta^{(r)}$ which is dual to σ. Then $v\not\in r_1H_1$. Pick $r_1'\gg r_1$ such that $r_1'\Delta_1\supseteq r_j\Delta_j$ for all j, $2\leq j\leq l$, and set $r':=(r_1',r_2,\dots,r_k)$. Then v is a vertex of $\Delta^{(r')}$ as well, and the facets of $\Delta^{(r)}$ containing v coincide with the facets of $\Delta^{(r')}$ containing v. It follows that σ is also a cone of $\mathcal{N}_{\Delta^{(r')}}$. But $\Delta^{(r')} = \bigcap_{i=2}^{l} r_i\Delta_i$ and $l(r') = l - 1$, so that, by induction, $\mathcal{N}_\Delta$ is a subdivision of $\mathcal{N}_{\Delta^{(r')}}$. The claim is then true for σ.

Case 2: $\boldsymbol{\eta_i \in \sigma}$ for all $\boldsymbol{i}$, $\boldsymbol{1 \leq i \leq l}$. Let σ' be an arbitrary full dimensional cone in $\mathcal{N}_{\Delta^{(r)}}$ such that $\sigma' \neq \sigma$. Then there exists $i(\sigma')$ such that $\eta_{i(\sigma')} \notin \sigma'$. The argument of Case 1 applied to σ' then shows that for every cone τ of $\mathcal{N}_\Delta$, if $\tau \cap \sigma' \neq \emptyset$, then $\tau \subseteq \sigma'$. But then σ must have the same property as well. ☐

The claim implies that $\mathcal{N}_\Delta$ is a refinement of $\mathcal{N}_{\Delta^{(r)}}$. The lemma then follows from fact 2 above. ☐

Corollary 6.8. *Let $\eta_1, \ldots, \eta_k$ be positive weighted degrees on $B := \mathbb{K}[x_1, \ldots, x_n]$ and $\eta := \max\{\eta_1, \ldots, \eta_k\}$. Let η_{diag} be the degree-like function corresponding to the diagonal embedding of $Y := \mathbb{K}^n$ into $Y^{\eta_1} \times \cdots \times Y^{\eta_k}$. Then there exists a morphism $Y^{\eta_{diag}} \to Y^\eta$ which is identity on Y.*

Proof. For each j, $1 \leq j \leq k$, Y^{η_j} is isomorphic to the toric variety X_{Δ_j} corresponding to a simplex Δ_j which satisfies the same property as of the hypothesis of lemma 6.7. Then Y^η is isomorphic to the toric variety corresponding to $\Delta_1 \cap \cdots \cap \Delta_k =: \Delta^{(1,\ldots,1)}$ (in the notation of lemma 6.7). Finally, fact 3 implies that $Y^{\eta_{\mathrm{diag}}} \cong X_\Delta$, where $\Delta := \Delta_1 + \cdots + \Delta_k$. The claim now follows directly from lemma 6.7. ☐

Now we are ready to state and prove the main result of this section:

Theorem 6.9. *Let X be an affine variety and $\delta_1, \ldots, \delta_k$ be finitely generated complete semidegrees on $A := \mathbb{K}[X]$. Define $\delta := \max\{\delta_1, \ldots, \delta_k\}$. If $\dim X = 2$, then δ is finitely generated and X^δ is isomorphic to the normalization at infinity (with respect to X) of the closure of the diagonal image of X in $X^{\delta_1} \times \cdots \times X^{\delta_k}$.*

Proof. W.l.o.g. assume that $\delta = \max\{\delta_1, \ldots, \delta_k\}$ is the minimal presentation of δ, i.e. for all j, $1 \leq j \leq k$, there exist $f_j \in A$ such that $\delta_j(f_j) > \delta_i(f_j)$ for all $i \neq j$, $1 \leq i \leq k$. Pick additional elements $f_{k+1}, \ldots, f_l \in A$ if necessary so that for every j, $1 \leq j \leq k$, A^{δ_j} is generated as a $\mathbb{K}$-algebra by $\{(1)_1\} \cup \{(f_j)_{\delta_j(f_j)} : 1 \leq j \leq l\}$.

Let $Y := \mathbb{K}^l$ with coordinates $y_1, \ldots, y_l$. Embed X into Y via the map $\phi : x \mapsto (f_1(x), \ldots, f_l(x))$. For each j, $1 \leq j \leq k$, let η_j be the weighted degree on $B := \mathbb{K}[y_1, \ldots, y_l]$ which assigns weights $\delta_j(f_i)$ to y_i for $1 \leq i \leq l$. Define $\eta := \max\{\eta_j : 1 \leq j \leq k\}$ and δ' to be the degree-like function on A induced by η, i.e.

$$\delta'(f) := \min\{\eta(F) : F \in B,\ F|_X = f\}$$

for all $f \in A$.

An application of corollary 6.8 to Y and η_j's imply that for each j, $1 \leq j \leq k$, there is a commuting system of morphisms of algebraic varieties as follows:

$$\begin{tikzcd}
Y^{\eta_1} \times \cdots \times Y^{\eta_k} \arrow[d, "\pi_j"] & Y^{\eta_{\mathrm{diag}}} \arrow[l, hook'] \arrow[dl] \arrow[d, "\rho"] \\
Y^{\eta_j} & Y^{\eta}
\end{tikzcd} \tag{11}$$

where π_j is the projection onto the j-th coordinate and ρ is the morphism provided by corollary 6.8. The construction of η_j's implies that for each j, $1 \leq j \leq k$, X^{δ_j} is isomorphic to the closure in Y^{η_j} of the image of X under the embedding $X \overset{\phi}{\hookrightarrow} Y \hookrightarrow Y^{\eta_j}$. Let δ_{diag} be the degree-like function corresponding to the diagonal embedding of X into $X^{\delta_1} \times \cdots \times X^{\delta_k}$. It then follows that $X^{\delta_{\text{diag}}}$ is isomorphic to the closure in $Y^{\eta_{\text{diag}}}$ of the image of X under the embedding $X \overset{\phi}{\hookrightarrow} Y \hookrightarrow Y^{\eta_{\text{diag}}}$. Similarly, $X^{\delta'}$ is isomorphic to the closure in Y^{η} of the image of X under the embedding $X \overset{\phi}{\hookrightarrow} Y \hookrightarrow Y^{\eta}$. Let $\tilde{\delta}_{\text{diag}}$ and $\tilde{\delta}'$ be *normalizations* of respectively δ_{diag} and δ', i.e.

$$\tilde{\delta}_{\text{diag}}(f) = e_{\text{diag}} \lim_{m\to\infty} \frac{\delta_{\text{diag}}(f^m)}{m}$$
$$\tilde{\delta}'(f) = e' \lim_{m\to\infty} \frac{\delta'(f^m)}{m}$$

for all $f \in A$ and suitable integers e_{diag} and e' to ensure $\tilde{\delta}_{\text{diag}}$ and $\tilde{\delta}'$ are integer valued. Then the morphisms in diagram (11) induces the following (commuting) systems of morphisms for each j, $1 \leq j \leq k$:

$$\begin{tikzcd}
X^{\delta_1} \times \cdots \times X^{\delta_k} \arrow[d, "\pi_j"] & X^{\delta_{\text{diag}}} \arrow[l, hook'] \arrow[d, "\rho"] \arrow[dl] & X^{\tilde{\delta}_{\text{diag}}} \arrow[l, "\phi_{\text{diag}}"'] \arrow[d, "\tilde{\rho}"] \\
X^{\delta_j} & X^{\delta'} & X^{\tilde{\delta}'} \arrow[l, "\phi'"']
\end{tikzcd} \tag{12}$$

where ϕ_{diag} and ϕ' are the natural finite maps associated with normalization of degree-like functions and $\tilde{\rho}$ is the (unique) lift of ρ to $X^{\tilde{\delta}'}$ (which exists because of the universal property of the normalizing subdegree). According to corollary 6.4, there exist positive rational numbers $r_1, \ldots, r_k$ such that $\tilde{\delta}_{\text{diag}}$ has minimal presentation $\tilde{\delta}_{\text{diag}} = \max\{r_1\delta_1, \ldots, r_k\delta_k\}$. Applying corollary 6.3 to $\tilde{\rho} : X^{\tilde{\delta}_{\text{diag}}} \to X^{\tilde{\delta}'}$ then yields that $\tilde{\delta}' = \max\{s_1\delta_1, \ldots, s_k\delta_k\}$ for some $s_1, \ldots, s_k \in \mathbb{Q}_+$. W.l.o.g. assume that the *minimal presentation* of $\tilde{\delta}'$ is $\tilde{\delta}' = \max\{s_1\delta_1, \ldots, s_{k'}\delta_{k'}\}$ for some $k' \leq k$.

Claim 6.9.1. *$k' = k$ and $s_1 = s_2 = \cdots = s_k = e'$.*

Proof. It follows from the definition of δ' that for all $f \in A$ and j, $1 \leq j \leq k$, $\delta'(f) \geq \delta_j(f)$, which implies that

$$\tilde{\delta}'(f) = e' \lim_{m\to\infty} \frac{\delta'(f^m)}{m} \geq e' \lim_{m\to\infty} \frac{\delta_j(f^m)}{m} = e' \lim_{m\to\infty} m\frac{\delta_j(f)}{m} = e'\delta_j(f). \tag{13}$$

It follows that

$$s_j \geq e' \text{ for all } j,\ 1 \leq j \leq k'. \tag{14}$$

Moreover, (13) implies that $\tilde{\delta}'(f_j) \geq e'\delta_j(f_j)$. On the other hand, for all j and m such that $1 \leq j \leq k$ and $m \geq 0$,

$$\delta'(f_j^m) \leq \eta(y_j^m) = m\eta(y_j) = m\delta_j(f_j),$$

so that $\tilde{\delta}'(f_j) \leq e'\delta_j(f_j)$. Therefore

$$\tilde{\delta}'(f_j) = e'\delta_j(f_j) \text{ for all } j,\ 1 \leq j \leq k. \tag{15}$$

Identities (15) and (14) together imply that

$$s_j = e' \text{ for all } j,\ 1 \leq j \leq k'. \tag{16}$$

This in turn implies that $k' = k$. Indeed, if $k' < k$, then

$$\begin{aligned}\tilde{\delta}'(f_{k'+1}) &= e'\delta_{k'+1}(f_{k'+1}) \text{ (due to (15))} \\ &> e'\delta_j(f_{k'+1}) \text{ for all } j,\ 1 \leq j \leq k' \text{ (due to the construction of } f_j\text{'s)} \\ &= \tilde{\delta}'(f_{k'+1}) \text{ (due to (16)).}\end{aligned}$$

This contradiction shows that $k' = k$ and completes the proof of the claim. ☐

Claim 6.9.1 implies that $\tilde{\delta}' = e'\delta$. Since $\tilde{\delta}'$ is finitely generated (theorem 5.12) and δ is integral over $\tilde{\delta}'$, it follows that δ is also finitely generated, which proves the first assertion of the theorem. Moreover, the orders of vanishing along the components of the curves at infinity on $X^{\tilde{\delta}_{\text{diag}}}$ and $X^{\tilde{\delta}'}$ are the same (both being $\{\frac{\delta_j}{d_j} : 1 \leq j \leq k\}$, where $d_j := \gcd\{\delta_j(f) : f \in A\}$, $1 \leq j \leq k$). Therefore proposition 6.5 implies that $X^{\tilde{\delta}_{\text{diag}}} \cong X^{\tilde{\delta}'} \cong X^{\delta}$, and this completes the proof of the theorem. ☐

Remark 6.10. A necessary condition for the existence of complete semidegrees on A is that the only invertible regular functions on X are the constants. Indeed, if there is $f \in A \setminus \mathbb{K}$ such that f^{-1} is also in A, then for every semidegree η on A, either $\eta(f)$ or $\eta(f^{-1}) = -\eta(f)$ is negative, or both of them are zero; in particular, η is *not* complete. It follows e.g. that there is no complete semidegree on the coordinate ring of $(\mathbb{K}^*)^n$, even though there are lots on the coordinate ring of $\mathbb{K}^n$.

6.3 The matrix of linking numbers at infinity is invertible

Let $\mathcal{I} := \{\delta_1, \ldots, \delta_k\}$ be a finite collection of complete semidegrees on A and $L_{\mathcal{I}}$ be the $k \times k$ *matrix of linking numbers at infinity* of δ_j's: i.e. the entries of $L_{\mathcal{I}}$ are $l_{ij} := l_\infty(\delta_i, \delta_j)$, $1 \leq i, j \leq k$, where, as in subsection 5.2,

$$l_\infty(\delta_i, \delta_j) := \max\left\{\frac{\delta_j(f)}{\delta_i(f)} : f \in A,\ \delta_i(f) > 0\right\}.$$

A necessary condition for the invertibility of $L_{\mathcal{I}}$ is that δ_i and δ_j are *not* proportional for all $i \neq j$, $1 \leq i, j \leq k$. Also recall that, as we remarked in subsection 1.2, if $k = 2$ then the preceding necessary condition is also sufficient for the invertibility of $L_{\mathcal{I}}$. Our final result will be to show that if $\dim A = 2$, then the same is true for arbitrary k. We start with a rather technical lemma.

Lemma 6.11. *Let X be an affine surface, $\delta_1, \ldots, \delta_k$ be mutually non-proportional complete semidegrees on $A := \mathbb{K}[X]$ and $\bar{X}$ be the closure of the diagonal embedding of X into $X^{\delta_1} \times \cdots \times X^{\delta_k}$. Then for all $s_1, \ldots, s_k \in \mathbb{N}$, there exists a subdegree δ on A such that*

1. *X^δ is isomorphic to the normalization at infinity of $\bar{X}$, and*
2. *for all $d \geq 0$, the d-uple divisor at infinity on X^δ is:*

$$D^\delta_{d,\infty} = m_d \sum_{i=1}^{k} \phi_i^*(D^{\delta_i}_{s_i,\infty}),$$

where $m_d \in \mathbb{Q}_+$ and for each i, $1 \leq i \leq k$, ϕ_i is the composition of the projection $\pi_i : X^{\delta_1} \times \cdots \times X^{\delta_k} \to X^{\delta_i}$ with the normalization-at-infinity map $\phi : X^\delta \to \bar{X}$.

Proof. Fix positive integers $s_1, \ldots, s_k$. Pick $q \geq 0$ such that for all i, $1 \leq i \leq k$, the qs_i-uple embedding maps X^{δ_i} isomorphically onto a subvariety V_i of $\mathbb{P}^{l_i}$, $l_i \in \mathbb{N}$. For all i, $1 \leq i \leq k$, let the homogeneous coordinates of $\mathbb{P}^{l_i}$ be $[z_{i0} : \cdots : z_{il_i}]$; without loss of generality we may assume that the restriction of the divisor of z_{i0} to X^{δ_i} is the divisor of $(1)_{qs_i}$, which is precisely $D^{\delta_i}_{qs_i,\infty}$. Let $l := \prod_{i=1}^k (l_i + 1) - 1$. The Segre embedding $s : \mathbb{P}^{l_1} \times \cdots \times \mathbb{P}^{l_k} \hookrightarrow \mathbb{P}^l$ induces an isomorphism ψ of $\bar{X}$ with a subvariety V of $\mathbb{P}^l$. Let δ_{diag} be the degree-like function on A induced by ψ.

The Segre mapping s maps $z := ([z_{10} : \cdots : z_{1l_1}], \ldots, [z_{k0} : \cdots : z_{kl_k}])$ to the point $s(z)$ whose homogeneous coordinates are monomials of degree k of the form $z_{1j_1} z_{2j_2} \cdots z_{kj_k}$ where $0 \leq j_i \leq l_i$ for each i, $1 \leq i \leq k$. Let $d \geq 0$. Then the d-uple divisor at infinity $D^{\delta_{\text{diag}}}_{d,\infty}$ on $X^{\delta_{\text{diag}}}$ is precisely the divisor of $(z_{10} z_{20} \cdots z_{k0})^d$. It follows that

$$D^{\delta_{\text{diag}}}_{d,\infty} = d \sum_{i=1}^k \operatorname{div}_{X^{\delta_{\text{diag}}}}(z_{i0}) = d \sum_{i=1}^k \pi_i^*(\operatorname{div}_{X^{\delta_i}}(z_{i0})) = d \sum_{i=1}^k \pi_i^*(D^{\delta_i}_{qs_i,\infty}). \tag{17}$$

Let $\tilde{\delta}_{\text{diag}}$ be a normalization of δ_{diag}. Recall from the construction of the normalization map (from the proof of theorem 5.12) that the map $\phi : X^\delta \to X^{\delta_{\text{diag}}}$ is induced by an inclusion $A^{e\delta_{\text{diag}}} \subseteq A^{\tilde{\delta}_{\text{diag}}}$ for some $e \geq 1$. It follows that $\phi^*(D^{\delta_{\text{diag}}}_{d,\infty}) = \phi^*(D^{e\delta_{\text{diag}}}_{ed,\infty}) = \phi^*(\operatorname{div}_{X^{e\delta_{\text{diag}}}}((1)_{ed}) = \operatorname{div}_{X^{\tilde{\delta}_{\text{diag}}}}((1)_{ed}) = D^{\tilde{\delta}_{\text{diag}}}_{ed,\infty}$. Combining the preceding equalities with (17), we see that $D^{\tilde{\delta}_{\text{diag}}}_{d,\infty} = \frac{d}{e} \sum_{i=1}^k (\pi_i \circ \phi)^*(D^{\delta_i}_{qs_i,\infty}) = \frac{dq}{e} \sum_{i=1}^k \phi_i^*(D^{\delta_i}_{s_i,\infty})$. Therefore, taking $\delta := \tilde{\delta}_{\text{diag}}$ and $m_d := \frac{dq}{e}$ completes the proof of the lemma. □

Proposition 6.12. *Let X be an affine variety and $\mathcal{I} := \{\delta_1, \ldots, \delta_k\}$ be a finite collection of mutually non-proportional complete semidegrees on the coordinate ring A of X. If $\dim X = 2$, then the matrix $L_{\mathcal{I}}$ of linking numbers at infinity of δ_j's is invertible.*

Proof. At first note that replacing any of the δ_i's by $s_i \delta_i$ for some $s_i \in \mathbb{Q}_+$ does not affect the invertibililty of $L_{\mathcal{I}}$. Therefore, without loss of generality we may assume that $d_i := \gcd\{\delta_i(f) : f \in A\} = 1$ for all i, $1 \leq i \leq k$. Assume contrary to the claim that $L_{\mathcal{I}}$ is not invertible. Then, renumbering δ_j's and replacing $\mathcal{I}$ by one of its subsets if necessary, we may assume that there are *positive* integers $a_1, \ldots, a_p, a'_1, \ldots, a'_q$, such that $p + q = k$ and $\sum_{i=1}^p a_i l_i = \sum_{j=1}^q a'_j l_{p+j}$, where l_j is the j-th row vector of $L_{\mathcal{I}}$, $1 \leq j \leq k$. Let $\bar{X}$ (resp. $\bar{X}'$) be the closure of the image of the diagonal embedding of X into $X^{\delta_1} \times \cdots \times X^{\delta_p}$ (resp. $X^{\delta_{p+1}} \times \cdots \times X^{\delta_{p+q}}$). According to lemma 6.11, there is a subdegree δ on A such that X^δ is isomorphic to the normalization at infinity of $\bar{X}$ and for all $d \geq 0$,

$$D^\delta_{d,\infty} = m_d \sum_{i=1}^p \phi_i^*(D^{\delta_i}_{a_i,\infty})$$

where $m_d \in \mathbb{Q}_+$ and for each i, $1 \leq i \leq p$, ϕ_i is the composition of the projection $\pi_i : X^{\delta_1} \times \cdots \times X^{\delta_p} \to X^{\delta_i}$ with the natural map $X^\delta \to \bar{X}$. Similarly, there is a subdegree δ' on

A such that $X^{\delta'}$ is isomorphic to the normalization at infinity of $\bar{X}'$ and for all $d \geq 0$,

$$D^{\delta'}_{d,\infty} = m'_d \sum_{j=1}^{q} {\phi'_j}^{*}(D^{\delta_{p+j}}_{a'_j,\infty})$$

where $m'_d \in \mathbb{Q}_+$ and for all j, $1 \leq j \leq q$, ϕ'_j is the composition of the projection $\pi'_j : X^{\delta_{p+1}} \times \cdots \times X^{\delta_{p+q}} \to X^{\delta_{p+j}}$ with the natural map $X^{\delta'} \to \bar{X}'$.

Let δ_{diag} be a degree-like function on A corresponding to the diagonal embedding of X into $X^{\delta_1} \times \cdots \times X^{\delta_k}$. Let $\tilde{\delta}_{\text{diag}}$ be a normalization of δ_{diag}. According to corollary 6.4, the minimal presentation of δ_{diag} is $\delta_{\text{diag}} = \max\{r_1\delta_1, \ldots, r_k\delta_k\}$ for some $r_1, \ldots, r_k \in \mathbb{N}$. The natural projection of $X^{\delta_{\text{diag}}}$ in the first p-coordinates induces a morphism $\psi : X^{\delta_{\text{diag}}} \to \bar{X}$. Due to the universal property of normalization at infinity, ψ lifts to a morphism $\tilde{\psi} : X^{\tilde{\delta}_{\text{diag}}} \to X^{\delta}$. Then for all $d \geq 0$,

$$\begin{aligned}
\tilde{\psi}^{*}(D^{\delta}_{d,\infty}) &= m_d \sum_{i=1}^{p} (\phi_i \circ \tilde{\psi})^{*}(D^{\delta_i}_{a_i,\infty}) \\
&= m_d \sum_{i=1}^{p} \sum_{m=1}^{k} a_i \frac{l^{\phi_i \circ \tilde{\psi}}_{\infty}(\delta_i, r_m\delta_m)}{r_m} [X^{\tilde{\delta}_{\text{diag}}}_{\infty,m}] \text{ (according to corollary 5.19)} \\
&= m_d \sum_{i=1}^{p} \sum_{m=1}^{k} a_i l_{im} [X^{\tilde{\delta}_{\text{diag}}}_{\infty,m}],
\end{aligned}$$

where for each j, $1 \leq j \leq k$, $X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}$ is the component of the hypersurface at infinity of $X^{\tilde{\delta}_{\text{diag}}}$ corresponding to $r_j\delta_j$. Similarly, there is a morphism $\tilde{\psi}' : X^{\tilde{\delta}_{\text{diag}}} \to X^{\delta'}$ which is identity on X and

$${\tilde{\psi}'}^{*}(D^{\delta'}_{d,\infty}) = m'_d \sum_{j=1}^{q} \sum_{m=1}^{k} a'_j l_{p+j,m} [X^{\tilde{\delta}_{\text{diag}}}_{\infty,m}].$$

Fix $d \geq 1$ and $c, c' \geq 1$ such that $m_d c = m'_d c'$. The choice of a_i and a'_j's then implies that

$$\tilde{\psi}^{*}(D^{\delta}_{cd,\infty}) = {\tilde{\psi}'}^{*}(D^{\delta'}_{c'd,\infty}). \tag{$*$}$$

Now, according to corollary 6.4, there are $s_1, \ldots, s_p, s'_1, \ldots, s'_q \in \mathbb{N}$ such that the minimal presentations of δ and δ' are respectively $\delta = \max\{s_1\delta_1, \ldots, s_p\delta_p\}$ and $\delta' = \max\{s'_1\delta_{p+1}, \ldots, s'_q\delta_{p+q}\}$. Therefore, corollary 5.19 implies that

$$\begin{aligned}
\tilde{\psi}^{*}(D^{\delta}_{cd,\infty}) &= cd \sum_{j=1}^{k} \frac{l^{\tilde{\psi}}_{\infty}(\delta, r_j\delta_j)}{r_j} [X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}] \\
&= cd \sum_{j=1}^{k} l^{\tilde{\psi}}_{\infty}(\delta, \delta_j) [X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}] \\
&= cd \sum_{j=1}^{k} \max\left\{ \frac{\delta_j(f)}{\delta(f)} : f \in A,\ \delta(f) > 0 \right\} [X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}]
\end{aligned}$$

$$= cd\left(\sum_{j=1}^{p}\frac{1}{s_j}[X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}] + \sum_{j=1}^{q}\max\left\{\frac{\delta_{p+j}(f)}{\delta(f)} : f \in A,\ \delta(f) > 0\right\}[X^{\tilde{\delta}_{\text{diag}}}_{\infty,p+j}]\right).$$

Similarly,

$$\tilde{\psi}^*(D^{\delta'}_{c'd,\infty}) = c'd\left(\sum_{j=1}^{p}\max\left\{\frac{\delta_j(f)}{\delta'(f)} : f \in A,\ \delta'(f) > 0\right\}[X^{\tilde{\delta}_{\text{diag}}}_{\infty,j}] + \sum_{j=1}^{q}\frac{1}{s'_j}[X^{\tilde{\delta}_{\text{diag}}}_{\infty,p+j}]\right).$$

Identity $(*)$ then implies that

$$\begin{aligned}\frac{c}{c'} &= \max\left\{\frac{s_j\delta_j(f)}{\delta'(f)} : f \in A,\ \delta'(f) > 0\right\} \text{ for all } j,\ 1 \le j \le p, \text{ and}\\ \frac{c'}{c} &= \max\left\{\frac{s'_j\delta_{p+j}(f)}{\delta(f)} : f \in A,\ \delta(f) > 0\right\} \text{ for all } j,\ 1 \le j \le q.\end{aligned} \tag{$**$}$$

The first of the two identities of $(**)$ implies that $\frac{\delta(f)}{\delta'(f)} \le \frac{c}{c'}$ for all $f \in A$ such that $\delta(f) > 0$. Pick $f_1 \in A$ such that $\frac{s_1\delta_1(f_1)}{\delta'(f_1)} = \frac{c}{c'}$. Since $\frac{s_j\delta_j(f_1)}{\delta'(f_1)} \le \frac{c}{c'}$ for all j, $1 \le j \le p$, it follows that $\delta(f_1) = s_1\delta_1(f_1)$ and therefore $\frac{\delta(f_1)}{\delta'(f_1)} = \frac{c}{c'}$. Consequently,

$$\begin{aligned}\frac{c}{c'} &= \max\left\{\frac{\delta(f)}{\delta'(f)} : f \in A,\ \delta'(f) > 0\right\}, \text{ and similarly}\\ \frac{c'}{c} &= \max\left\{\frac{\delta'(f)}{\delta(f)} : f \in A,\ \delta(f) > 0\right\}.\end{aligned}$$

Since each of δ and δ' is non-negative and takes zero value only on $\mathbb{K}$, it follows that $\frac{\delta(f)}{\delta'(f)} = \frac{c}{c'}$ for *all* $f \in A$, or equivalently, $\delta = \frac{c}{c'}\delta'$. It follows that $\delta = \max\{s_1\delta_1, \ldots, s_p\delta_p\}$ and $\delta = \max\{\frac{cs'_1}{c'}\delta_{p+1}, \ldots, \frac{cs'_q}{c'}\delta_{p+q}\}$ are two different minimal presentations of δ. But this contradicts the uniqueness of the minimal presentation of a subdegree. Therefore $L_{\mathcal{I}}$ is invertible, as required. ☐

Corollary 6.13. *Let X be an affine surface with trivial Picard group and δ be a subdegree on $A := \mathbb{K}[X]$ with minimal presentation $\delta = \max\{\delta_1, \ldots, \delta_k\}$, where each δ_j is a complete semidegree, $1 \le j \le k$. Then* $\operatorname{Pic} X^\delta \cong \mathbb{Z}^k$.

Proof. Let $\mathcal{D}$ be the collection of Cartier divisors D on X^δ such that $\operatorname{Supp} D \subseteq X_\infty$. We claim that if D is a non-zero element of $\mathcal{D}$, then $D \neq 0 \in \operatorname{Pic} X^\delta$. Indeed, if $D = \operatorname{div}(f)$ for some $f \in A$, then f is invertible on X and therefore $f \in \mathbb{K}$ (see remark 6.10). Then D is necessarily the zero Cartier divisor, contradicting the assumption on D. It follows that there is an exact sequence of the form

$$0 \to \mathcal{D} \to \operatorname{Pic} X^\delta \to \mathcal{E} \to 0,$$

where $\mathcal{E}$ is the subgroup of $\operatorname{Pic} X$ comprising of the (linear equivalence classes) of the Cartier divisors on X which are restrictions of Cartier divisors on X^δ and $\operatorname{Pic} X^\delta \to \mathcal{E}$ is the restriction map. Since $\operatorname{Pic} X = 0$, it follows that $\mathcal{E} = 0$ and therefore $\operatorname{Pic} X^\delta \cong \mathcal{D}$. In particular,

$\operatorname{Rank}\operatorname{Pic} X^\delta = \operatorname{Rank}\mathcal{D} \leq k$.

Now recall (from theorem 6.9) that X^δ is the normalization at infinity of the closure of the diagonal image of X in $X^{\delta_1} \times \cdots \times X^{\delta_k}$ and therefore, for each i, $1 \leq i \leq k$, there is a natural morphism $\phi_i : X^\delta \to X^{\delta_i}$ which is identity on X. Pick a suitable $r \geq 1$ such that $D^{\delta_i}_{r,\infty}$ is a Cartier divisor (as opposed to a $\mathbb{Q}$-Cartier divisor) on X^{δ_i} for each i, $1 \leq i \leq k$. Then for each i, $1 \leq i \leq k$, $\phi_i^*(D^{\delta_i}_{r,\infty}) = r\sum_{j=1}^k \frac{l_{ij}}{d_j}[X^\delta_{\infty,j}] \in \mathcal{D}$, where l_{ij} are as in proposition 6.12 and $d_j := \gcd\{\delta_j(f) : f \in A\}$. Proposition 6.12 then implies that $\{\phi_i^*(D^{\delta_i}_{r,\infty}) : 1 \leq i \leq k\}$ is a $\mathbb{Q}$-linearly independent subset of $\mathcal{D}$, i.e. $\operatorname{Rank}\mathcal{D} \geq k$. It follows that $\operatorname{Rank}\mathcal{D} = k$ and the corollary follows. ☐

Example 6.14. Let $X = \mathbb{K}^2$, δ_1 be the usual degree on $A := \mathbb{K}[x_1, x_2]$ and δ_2 be the *iterated semidegree* on A defined in example 3.3. Recall that $\delta_2(x_1) = 3$, $\delta_2(x_2) = 3$ and $\delta_2(x_1^2 - x_2^3) = 1$. Let $\delta := \max\{\delta_1, \delta_2\}$. Then theorem 6.9 implies that δ is finitely generated and X^δ is isomorphic to the normalization at infinity of the closure of the diagonal image of X in $X^{\delta_1} \times X^{\delta_2}$. Moreover, it follows according to corollary 6.13 that $\operatorname{Pic} X^\delta \cong \mathbb{Z}^2$.

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